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On the uniqueness of minimal coupling in higher-spin gauge theory

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Abstract: We address the uniqueness of the minimal couplings between higher-spin fields and gravity. These couplings are cubic vertices built from gauge non-invariant connections that induce non-abelian deformations of the gauge algebra. We show that Fradkin-Vasiliev's cubic $2-s-s$ vertex, which contains up to $2s-2$ derivatives dressed by a cosmological constant Λ, has a limit where: (i) $\Lambda \to 0$; (ii) the spin-2 Weyl tensor scales *non-uniformly* with s; and (iii) all lower-derivative couplings are scaled away. For $s=3$ the limit yields the unique non-abelian spin $2-3-3$ vertex found recently by two of the authors, thereby proving the *uniqueness* of the corresponding FV vertex. We extend the analysis to $s=4$ and a class of spin $1-s-s$ vertices. The non-universality of the flat limit high-lightens not only the problematic aspects of higher-spin interactions with $\Lambda=0$ but also the strongly coupled nature of the derivative expansion of the fully nonlinear higher-spin field equations with $\Lambda \neq 0$, wherein the standard minimal couplings mediated via the Lorentz connection are *subleading* at energy scales $\sqrt{|\Lambda|} \ll E \ll M_{\mathrm{p}}$. Finally, combining our results with those obtained by Metsaev, we give the complete list of *all* the manifestly covariant cubic couplings of the form $1-s-s$ and $2-s-s$, in Minkowski background.

Keywords: Gauge Symmetry, BRST Symmetry.

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Contents

1. Introduction and overview

1.1 No-go and yes-go results for $\Lambda = 0$

From a general perspective it is a remarkable fact that the full gravitational couplings of lower-spin fields involve at most two derivatives in the Lagrangian. For spin $s \leqslant 1$ the *standard* covariantization scheme, wherein $\partial \to \nabla = \partial + \omega$ with ω being a torsion-constrained Lorentz connection, induces the "minimal coupling" $\int d^D x\, h_{\mu\nu} T^{\mu\nu}$ where $T^{\mu\nu}$ is the Belifante-Rosenfeld stress-tensor which is quadratic and contains up to two derivatives. Actually, for scalars, Maxwell fields and other Lorentz-invariant differential forms, the Lorentz covariantization is trivial and the coupling therefore involves no derivatives of the metric. It is also remarkable that the non-abelian cubic self-coupling of a spin-2 field contains only two derivatives.

Turning to gauge fields with $s > 2$ and considering $2 - s - s$ couplings in an expansion around flat spacetime, the standard scheme breaks down as has been known for a long time [1 − 3]. These no-go results have recently been strengthened in [4, 5] following a light-cone method and in [6] with S-matrix tools. More interestingly, in the works [4, 7, 5] some *yes-go* results have been obtained. In the specific case of $s = 3$, the work [7] provides a manifestly covariant *non-standard* four-derivative vertex associated with a *nonabelian* deformation of the gauge algebra. These yes-go results suggest a class of minimal nonabelian[1] non-standard vertices containing $2s - 2$ derivatives. We wish to emphasize that the existence of cubic couplings containing $2s-2$ derivatives was explicitly shown in [4, 5] although the light-cone gauge method used therein does not exhibit the nature of the gauge algebra and does not readily allow for the explicit construction of the corresponding covariant vertices. The results are nonetheless remarkable in that they show the existence of only a few non-trivial cubic vertices of the general form $s - s' - s''$ for massive and massless fields (bosonic and fermionic) in flat space of arbitrary dimension $D > 3$. In the case of integer spins, the possible vertices have $s + s' + s'' - p$ derivatives where $p = 0, 2, \ldots, 2\min(s, s', s'')$.

In the specific massless $2 - 3 - 3$ case, using the BRST-BV cohomological methods of [9, 10], the vertex of [7] was shown to be *unique* among the class of vertices that: (i) contain a *finite* number of derivatives; (ii) manifestly preserve Poincaré invariance and (iii) induce a *nonabelian* deformation of the gauge algebra. This uniqueness result relies on the fact that other candidate nonabelian deformations cannot be "integrated" cohomologically to gauge transformations and vertices. We have managed to push the uniqueness analysis to the case of $s = 4$ and the unique $2 - 4 - 4$ nonabelian vertex is presented in section 4 together with its corresponding gauge algebra and transformations.

We also extend the results of [7] with the cohomological proof in section 5 that the standard two-derivative minimal couplings $2 - s - s$ are inconsistent, thereby providing an alternative proof for the results recently obtained in [4, 5] following light-cone methods and in [6] following S-matrix methods. In the same section 5, combining the cohomological approach with the light-cone results of Metsaev [4, 5], we show that there exists *only one*

[1] We consider only couplings that truly deform the initial abelian gauge algebra into a nonabelian one, similarly to what happens when coupling $N^2 - 1$ Maxwell fields in order to obtain the Yang-Mills SU(N) theory. Interesting results and references on abelian couplings can be found in the review [8].

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nonabelian $2-s-s$ coupling, which contains $2s-2$ derivatives and must be the flat limit of the well-known nonabelian Fradkin-Vasiliev vertex [11, 12] in AdS, as we verify explicitly for $s=3$. There also exist two abelian covariant $2-s-s$ vertices containing $2s+2$ and $2s$ derivatives. Their existence was first found in [4], and we exhibit them here explicitly in their covariant form. The $(2s+2)$-derivative vertex is of the Born-Infeld type, whereas the $2s$-derivative vertex exists only for $D\geqslant 5$ and is gauge invariant up to a total derivative. These three vertices, with $2s-2$, $2s$ and $2s+2$ derivatives, thus exhaust the possibilities of manifestly Lorentz-covariant $2-s-s$ couplings in flat space.

We begin in section 3 by examining the simpler case of $1-s-s$ vertices. We build explicitly the unique, nonabelian $1-s-s$ coupling, which has $2s-1$ derivatives, together with the only abelian $1-s-s$ vertex, which as $2s+1$ derivatives, thereby completing the list of all possible nontrivial, manifestly covariant, $1-s-s$ couplings. Again, by the uniqueness of the nonabelian vertex, we know that it is the flat limit of the corresponding AdS Fradkin-Vasiliev (FV) vertex [11, 12].

1.2 The Fradkin-Vasiliev cancelation mechanism for $\Lambda\neq 0$

Under the assumptions that the cosmological constant vanishes and that the Lagrangian contains at most two derivatives, the standard covariantization of Fronsdal's action leads to an inconsistent cubic action of the form[2]

$$S^{\Lambda=0}_{2ss}[g,\phi]=\frac{1}{\ell_p^{D-2}}\int\left(R+G+\frac{1}{2}W_{\mu\nu\rho\sigma}\beta^{\mu\nu,\rho\sigma}_{(2)}(\phi^{\otimes 2})\right), \tag{1.1}$$

where ℓ_p is the Planck length, $\int=\int d^Dx\sqrt{-g}$, the spin-s kinetic term[3] $G=\frac{1}{2}\phi^{\mu(s)}G_{\mu(s)}$ with the Einstein-like self-adjoint operator[4]

$$G_{\mu(s)}=F_{\mu(s)}-\frac{s(s-1)}{4}g_{\mu(2)}F'_{\mu(s-2)}\,, \tag{1.2}$$

$$F_{\mu(s)}=\nabla^2\phi_{\mu(s)}-s\nabla_{\mu_1}\nabla\cdot\phi_{\mu(s-1)}+\frac{s(s-1)}{2}\nabla_{\mu_1}\nabla_{\mu_2}\phi'_{\mu(s-2)}\,, \tag{1.3}$$

the covariantized Fronsdal field strength. The symbol $\beta_{(2)}$ denotes a dimensionless symmetric bilinear form, $W_{\mu\nu\rho\sigma}$ is the spin-2 Weyl tensor, and $(\ell_p)^2W$ and ϕ are assumed to be weak fields. A quantity $\mathscr{O}$ has a regular weak-field expansion if $\mathscr{O}=\sum_{n=n(\mathscr{O})}^{\infty}\overset{(n)}{\mathscr{O}}$ where $\overset{(n)}{\mathscr{O}}$ scales like g^n if the weak fields are rescaled by a constant factor g, and we shall refer to $\overset{(n)}{\mathscr{O}}$ as being of nth in weak fields, or equivalently, as being of order $n-n(\mathscr{O})$ in the g

[2]We use mostly positive signature and $R=g^{\mu\rho}g^{\nu\sigma}R_{\mu\nu\rho\sigma}$. The Fierz-Pauli action $\int d^Dx(-\frac{1}{2}\partial^\mu h^{\rho\sigma}\partial_\mu h_{\rho\sigma}+\cdots)$ is recuperated modulo boundary terms from $\frac{1}{(\ell_p)^{D-2}}\int d^Dx\sqrt{-g}R(g)$ upon substituting $g_{\mu\nu}=\eta_{\mu\nu}+\sqrt{2}(\ell_p)^{\frac{D-2}{2}}h_{\mu\nu}$.

[3]The initial choice of free kinetic terms affects the classical anomaly and the final form of anomaly cancelation terms.

[4]Repeated indices distinguished by sub-indexation are implicitly symmetrized, so that e.g. $g_{\mu(2)}F'_{\mu(s-2)}\equiv g_{(\mu_1\mu_2}F'_{\mu_3\ldots\mu_s)}$ where $(\cdots)$ denotes symmetrization with strength one. Moreover, $\nabla\cdot V_{\mu(s-1)}\equiv\nabla^\nu V_{\nu\mu(s-1)}$ and $V'_{\mu(s-2)}\equiv g^{\nu\rho}V_{\nu\rho\mu(s-2)}$.

expansion. Under the spin-s gauge transformation $\delta_\varepsilon\phi_{\mu(s)} = s\,\nabla_{\mu_1}\varepsilon_{\mu(s-1)} + R_{\mu(s)}[g_{\alpha\beta},\phi,\varepsilon]$ and $\delta_\varepsilon g_{\mu\nu} = R_{\mu\nu}[g_{\alpha\beta},\phi,\varepsilon]$, where ε is a weak traceless parameter and $R_{\mu(s)}$ and $R_{\mu(2)}$ are quadratic in weak fields, the variation of the action picks up the first-order contribution

$$\delta_\varepsilon \int G = \int W^{\mu\nu\rho\sigma}\mathscr{A}_{\mu\nu,\rho\sigma}(g_{\alpha\beta};\nabla\phi\otimes\varepsilon)\,, \tag{1.4}$$

where the bilinear form

$$\begin{aligned}\mathscr{A}_{\mu\nu,\rho\sigma} &= 2s(s-1)\mathbf{P}^W\Big[\nabla_\mu\phi_{\nu\sigma\tau(s-2)}\varepsilon_\rho{}^{\tau(s-2)}\\ &+(s-2)\big(\nabla_\sigma\phi'_{\nu\tau(s-3)}-\frac{1}{2}\nabla\cdot\phi_{\nu\sigma\tau(s-3)}+\frac{(s-3)}{4}\nabla_{\tau_1}\phi'_{\nu\sigma\tau(s-4)}\big)\varepsilon_{\mu\rho}{}^{\tau(s-3)}\Big]\,,\end{aligned} \tag{1.5}$$

that has been shown to be anomalous for $s=3$ [3] (for recent re-analysis see [7] and also [6] for an S-matrix argument) in the sense that it cannot be canceled by any choice of $\beta_{(2)}$ nor by abandoning the assumption that the Lagrangian contains at most two derivatives.

However, as first realized by Fradkin and Vasiliev [11], if both $\Lambda\neq 0$ and higher-derivative terms are added to the cubic part of the action, the analogous obstruction can be bypassed. In the weak-field expansion the resulting *minimal cubic action* reads

$$S^\Lambda_{2ss}[g,\phi] = \frac{1}{\ell_p^{D-2}}\int (R(g)-\Lambda+G_\Lambda) + \sum_{\substack{n=2\\ n\ even}}^{n_{\min}(s)}\frac{1}{\ell_p^{D-2}}\int V^{(n)}_\Lambda(2,s,s)\,, \tag{1.6}$$

$$V^{(n)}_\Lambda(2,s,s) = \frac{1}{2\lambda^{n-2}}\sum_{p+q=n-2} W_{\mu\nu\rho\sigma}\beta^{\mu\nu,\rho\sigma}_{(n);p,q}(\nabla^p\phi\otimes\nabla^q\phi)\,, \tag{1.7}$$

with $\lambda^2\equiv-\frac{\Lambda}{(D-1)(D-2)}$, and $G_\Lambda = G-\frac{1}{2}\lambda^2 M_s^2(\phi^{\otimes 2})$ with G defined in (1.2), and $M_s^2(\phi^{\otimes 2}) = m_s^2\phi^2+m'^2_s\phi'^2$. At zeroth order, the spin-s gauge invariance requires the critical masses [13]

$$m_s^2 = s^2+(D-6)s-2D+6\,,\qquad m'^2_s = -\frac{1}{2}s(s-1)\left[s^2+(D-4)s-D+1\right]\,. \tag{1.8}$$

Up to first order, the invariance uses zeroth order on-shell conditions

$$[\nabla_\mu,\nabla_\nu]V_\rho = R_{\mu\nu\rho}{}^\sigma V_\sigma \approx 2\lambda^2 g_{\rho[\nu}V_{\mu]}+W_{\mu\nu\rho}{}^\sigma V_\sigma\,,\qquad \nabla^\mu W_{\mu\nu,\rho\sigma}\approx 0\,, \tag{1.9}$$

$$R_{\mu\nu}-\frac{1}{2}(R-\Lambda)g_{\mu\nu}\approx 0\,,\qquad F_{\mu(s)}-\lambda^2\left(m_s^2\phi_{\mu(s)}+s(s-1)g_{\mu(2)}\phi'_{\mu(s-2)}\right)\approx 0\,, \tag{1.10}$$

where $F_{\mu(s)}$ is defined in (1.3). At first order, the classical anomaly $\int W\mathscr{A}$, which is independent of λ, is accompanied by two types of λ-independent counter terms, namely $\delta_\varepsilon\int V^{(2)}_\Lambda$ plus the contributions to $\delta_\varepsilon\int V^{(4)}_\Lambda$ from the constant-curvature part of $[\nabla,\nabla]$, that can be arranged to cancel the anomaly at order λ^0. At order λ^{-2}, the remaining terms in $\delta_\varepsilon\int V^{(4)}_\Lambda$ can be canceled against order λ^{-2} contributions from $\delta_\varepsilon\int V^{(6)}_\Lambda$ and so on, until the procedure terminates at the *top vertex* $V^{\rm top}_\Lambda(2,s,s)=V^{(n_{\min}(s))}_\Lambda$ that: (i) is *weakly gauge invariant* up to total derivatives and terms that are of lower order in λ; and (ii) contains a total number of derivatives given by

$$n_{\min}(s) = 2s-2\,. \tag{1.11}$$

Counting numbers of derivatives, there is a *gap* between the top vertex and the *tail* of Born-Infeld-like non-minimal cubic vertices, which is *a priori* of the form

$$S^{\rm nm}_{2ss;\Lambda} = \sum_{n=0}^{\infty} \frac{1}{(\ell_p)^{D-2} 2\lambda^{2(n+s)}} \sum_{p+q=2n} \int W_{\mu\nu\rho\sigma} \gamma^{\mu\beta,\rho\sigma}_{(n);p,q} (\nabla^p C \otimes \nabla^q C) \,, \tag{1.12}$$

where $C_{\mu(s),\nu(s)}$ is the linearized spin-s Weyl tensor and $\gamma_{(n);p,q}$ are dimensionless bilinear forms. Adapting the flat-space result of [4] to constantly curved backgrounds suggests that, if the $\gamma_{(n);p,q}$ fall off with n sufficiently fast, then the couplings with $n \geqslant 1$ can be removed by a suitable, possibly non-local, field redefinition. More generally, turning to higher orders in the weak-field expansion, one may adopt the *canonical* frame of standard fields that by definition minimizes the maximal numbers of derivatives at each order.

The *existence* of at least one cancelation procedure has sofar been shown in the literature only for $D = 4, 5$ [11, 14, 15], following the existence of a more general minimal cubic action given within the frame-like formulation based on a nonabelian higher-spin Lie algebra extension $\mathfrak{h}$ of $\mathfrak{so}(D+1;\mathbb{C})$. The 4D action is a natural generalization of the MacDowell-Mansouri action for Λ-gravity. It is given by a four-form Lagrangian based on a bilinear form $< \cdot, \cdot >_{\mathfrak{h}}$ such that the resulting action: (i) contains at most 2 derivatives at second order in weak fields; (ii) propagates symmetric rank-s tensor gauge fields with $s \geqslant 1$ and critical mass; (iii) contains nonabelian $V^{(n)}_{\Lambda}(s, s', s'')$ vertices with $s, s', s'' \geqslant 1$ and $n \leqslant n_{\rm min}(s, s', s'')$. The 5D action shares the same basic features [14, 15]. The existence issue in $D > 5$ is open at present though all indications sofar hint at that the lower-dimensional cases do actually have a generalization to arbitrary D.

1.3 Recovering the metric-like FV 2-3-3 vertex

Apparently Fradkin and Vasiliev first found the gravitational coupling of the spin-3 field using the metric-like formalism without publishing their result (see [16] for an account). Later they obtained and published their (by now famous) result in the frame-like formalism in the general $2 - s - s$ case in $D = 4$ [11]. For the purpose of discussing the uniqueness of their result and its extension to D dimensions, we need the explicit form of the D-dimensional $2 - 3 - 3$ FV vertex. To this end, we work within the metric-like formulation and start from the free Lagrangian $\mathscr{L}_2 + \mathscr{L}_3$ where Fronsdal's Lagrangian for a symmetric rank-s tensor gauge field in AdS_D reads [17]

$$\begin{aligned}
-\frac{\mathscr{L}_s}{\sqrt{-\bar g}} &= \frac{1}{2}\,\overline{\nabla}_\mu \phi_{\alpha_1\dots\alpha_s} \overline{\nabla}^\mu \phi^{\alpha_1\dots\alpha_s} - \frac{1}{2}\, s \overline{\nabla}^\mu \phi_{\mu\alpha_1\dots\alpha_{s-1}} \overline{\nabla}_\nu \phi^{\nu\alpha_1\dots\alpha_{s-1}} \\
&\quad + \frac{1}{2}\, s(s-1) \overline{\nabla}_\alpha \phi'_{\beta_1\dots\beta_{s-2}} \overline{\nabla}_\mu \phi^{\mu\alpha\beta_1\dots\beta_{s-2}} - \frac{1}{4}\, s(s-1) \overline{\nabla}_\mu \phi'_{\alpha_1\dots\alpha_{s-2}} \overline{\nabla}^\mu \phi'^{\alpha_1\dots\alpha_{s-2}} \\
&\quad - \frac{1}{8} s(s-1)(s-2) \overline{\nabla}^\mu \phi'_{\mu\alpha_1\dots\alpha_{s-3}} \overline{\nabla}_\nu \phi'^{\nu\alpha_1\dots\alpha_{s-3}} \\
&\quad + \frac{1}{2}\lambda^2 \left[s^2 + (D-6)s - 2D + 6\right] \phi_{\alpha_1\dots\alpha_s} \phi^{\alpha_1\dots\alpha_s} \\
&\quad - \frac{1}{4}\lambda^2 s(s-1) \left[s^2 + (D-4)s - D + 1\right] \phi'_{\alpha_1\dots\alpha_{s-2}} \phi'^{\alpha_1\dots\alpha_{s-2}} \,,
\end{aligned} \tag{1.13}$$

given that $\overline{R}_{\alpha\beta\gamma\delta} = -\lambda^2(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\beta\gamma}\bar{g}_{\alpha\delta})$. We find, using the Mathematica package Ricci [18], that the 2-3-3 FV vertex is given by[5]

$$
\begin{aligned}
-\frac{\overset{(3)}{\mathscr{L}}_{\rm FV}}{\sqrt{-\bar{g}}} \approx & -\frac{11}{2}\, w_{\alpha\beta\gamma\delta}\, \phi^{\alpha\gamma}{}_{\mu}\phi^{\beta\delta\mu} + \frac{1}{(D-1)\lambda^2}\, w_{\alpha\beta\gamma\delta}\, \Big[2\, \phi'_{\mu}\overline{\nabla}^{(\beta}\overline{\nabla}^{\delta)}\phi^{\alpha\gamma\mu} + \phi^{\alpha\gamma}{}_{\mu}\overline{\nabla}^{(\delta}\overline{\nabla}^{\mu)}\phi'^{\beta} \\
& -3\, \phi'^{\alpha}\overline{\nabla}^{(\delta}\overline{\nabla}^{\mu)}\phi^{\beta\gamma}{}_{\mu} + 2\, \phi^{\alpha}{}_{\mu\nu}\overline{\nabla}^{(\delta}\overline{\nabla}^{\nu)}\phi^{\beta\gamma\mu} + \overline{\nabla}_{\mu}\phi^{\alpha\gamma\mu}\overline{\nabla}_{\nu}\phi^{\beta\delta\nu} \\
& -\phi^{\alpha\gamma\mu}\overline{\nabla}_{(\mu}\overline{\nabla}_{\nu)}\phi^{\beta\delta\nu} - 2\, \overline{\nabla}^{(\mu}\phi^{\nu)\alpha\gamma}\overline{\nabla}_{\mu}\phi^{\beta\delta}{}_{\nu} - 2\, \phi^{\alpha\gamma}{}_{\mu}\overline{\nabla}^{(\delta}\overline{\nabla}^{\nu)}\phi^{\beta\mu}{}_{\nu} \\
& +\phi'^{\alpha}\overline{\nabla}^{(\beta}\overline{\nabla}^{\delta)}\phi'^{\gamma} - \phi^{\alpha}{}_{\mu\nu}\overline{\nabla}^{(\beta}\overline{\nabla}^{\delta)}\phi^{\gamma\mu\nu}\Big]\,.
\end{aligned} \tag{1.14}
$$

The first term corrects the obstruction to the standard minimal scheme at the expense of introducing a new one that can be removed, however, by adding the above particular combination of two-derivative terms (involving only a subset of all possible tensorial structures as expected from the frame- like formulation). We stress again that the top vertex does not introduce any further obstructions, and that the vertex indeed exhibits the gap.

1.4 Non-uniform $\Lambda \to 0$ limits

Since for given s the derivative expansion of the minimal $2-s-s$ coupling terminates at the top vertex $V_{\Lambda}^{(2s-2)}(2,s,s)$, the cubic action S_{2ss}^{Λ} admits the *scaling limit*

$$\lambda = \varepsilon(\ell_p)^{-1}\,, \quad W = \varepsilon^{2s-4}\widetilde{W}\,, \qquad \varepsilon \to 0\,, \tag{1.15}$$

with evanescent piece $\widetilde{W}_{\mu\nu\rho\sigma}$ held fixed, so that $\widetilde{W}_{\mu\nu\rho\sigma}$ can be replaced by the linearized Weyl tensor $\widetilde{w}_{\mu\nu\rho\sigma}$ in the cubic vertices, resulting in the action

$$\widetilde{S}_{2ss}^{\Lambda=0}[g,\phi] = \frac{1}{\ell_p^{D-2}}\int d^Dx\sqrt{-g}\,\Big(R(g) + G_0 + \widetilde{V}_0^{(2s-2)}(2,s,s)\Big)\,, \tag{1.16}$$

$$\widetilde{V}_0^{(2s-2)}(2,s,s) = \frac{1}{2}\sum_{p+q=2s-4}\int \widetilde{w}_{\mu\nu\rho\sigma}\beta^{\mu\nu,\rho\sigma}_{(n);p,q}(\nabla^p\phi\otimes\nabla^q\phi)\,, \tag{1.17}$$

that is faithful up to cubic order in weak graviton and spin-s fields, and G_0 contains the connection ∇_0 obeying the flatness condition $[\nabla_0,\nabla_0]=0$.

Alternatively, one may first perturbatively expand the FV action around AdS and then take the $\Lambda \to 0$ limit as follows:

$$\lambda = \varepsilon\widetilde{\ell}_p^{-1}\,, \qquad \ell_p = \varepsilon^{\Delta_p}\widetilde{\ell}_p\,, \tag{1.18}$$

$$h_{\mu\nu} = \varepsilon^{\Delta_h}\widetilde{h}_{\mu\nu}\,, \qquad \phi_{\mu\nu\rho} = \varepsilon^{\Delta_\phi}\widetilde{\phi}_{\mu\nu\rho}\,, \qquad \varepsilon \to 0\,, \tag{1.19}$$

[5] We use conventions where $h_{\alpha\beta}$ and $\phi_{\alpha\beta\gamma}$ are dimensionless. The linearized spin-2 Weyl tensor $w_{\alpha\beta\gamma\delta} = s_{\alpha\beta\gamma\delta} - \frac{2}{D-2}(\bar{g}_{\alpha[\gamma}s_{\delta]\beta} - \bar{g}_{\beta[\gamma}s_{\delta]\alpha}) + \frac{2}{(D-1)(D-2)}\bar{g}_{\alpha[\gamma}\bar{g}_{\delta]\beta}s$, where $s_{\alpha\beta\gamma\delta} \equiv -\overline{\nabla}_{\gamma}\overline{\nabla}_{[\alpha}h_{\beta]\delta} + \overline{\nabla}_{\delta}\overline{\nabla}_{[\alpha}h_{\beta]\delta} + \lambda^2(\bar{g}_{\gamma[\alpha}h_{\beta]\delta} - \bar{g}_{\delta[\alpha}h_{\beta]\gamma})$ has the property that at zeroth order $\bar{g}^{\alpha\beta}s_{\alpha\gamma\beta\delta} \approx 0$ and $\overline{\nabla}^{\alpha}s_{\alpha\beta\gamma\delta} \approx 0$. The form of the 2-3-3 FV vertex given in (1.14) reflects the initial choice of free Lagrangian made in (1.13).

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with $\widetilde{\ell}_p$, $\widetilde{h}$ and $\widetilde{\phi}$ kept fixed and $\Delta_h = \Delta_\phi = 2(s-2)$ and $\Delta_p = \frac{4(s-2)}{D-2}$. The resulting flat-space 2-3-3 vertex reads

$$
\begin{aligned}
-\overset{(3)}{\mathscr{L}} = \frac{1}{D-1}\tilde{w}_{\alpha\beta\gamma\delta}\Big[& 2\,\widetilde{\phi}'_{\mu}\partial^\beta\partial^\delta\widetilde{\phi}^{\alpha\gamma\mu} + \widetilde{\phi}^{\alpha\gamma}{}_{\mu}\partial^\delta\partial^\mu\widetilde{\phi}'^{\beta} - 3\,\widetilde{\phi}'^{\alpha}\partial^\delta\partial^\mu\widetilde{\phi}^{\beta\gamma}{}_{\mu} \\
& +2\,\widetilde{\phi}^{\alpha}{}_{\mu\nu}\partial^{(\delta}\partial^{\nu)}\widetilde{\phi}^{\beta\gamma\mu} + \partial_\mu\widetilde{\phi}^{\alpha\gamma\mu}\partial_\nu\widetilde{\phi}^{\beta\delta\nu} - \widetilde{\phi}^{\alpha\gamma\mu}\partial_\mu\partial_\nu\widetilde{\phi}^{\beta\delta\nu} \\
& -2\,\partial^{(\mu}\widetilde{\phi}^{\nu)\alpha\gamma}\partial_\mu\widetilde{\phi}^{\beta\delta}{}_{\nu} - 2\,\widetilde{\phi}^{\alpha\gamma}{}_{\mu}\partial^\delta\partial^\nu\widetilde{\phi}^{\beta\mu}{}_{\nu} + \widetilde{\phi}'^{\alpha}\partial^\beta\partial^\delta\widetilde{\phi}'^{\gamma} - \widetilde{\phi}^{\alpha}{}_{\mu\nu}\partial^\beta\partial^\delta\widetilde{\phi}^{\gamma\mu\nu}\Big]
\end{aligned} \tag{1.20}
$$

where $\tilde{w}_{\alpha\beta\gamma\delta} = \tilde{K}_{\alpha\beta\gamma\delta} - \frac{2}{D-2}\left(\eta_{\alpha[\gamma}\tilde{K}_{\delta]\beta} - \eta_{\beta[\gamma}\tilde{K}_{\delta]\alpha}\right) + \frac{2}{(D-1)(D-2)}\,\eta_{\alpha[\gamma}\eta_{\delta]\beta}\tilde{K}$ with $\tilde{K}_{\alpha\beta\gamma\delta} = -\partial_\gamma\partial_{[\alpha}\tilde{h}_{\beta]\delta} + \partial_\delta\partial_{[\alpha}\tilde{h}_{\beta]\delta}$.

As discussed above, the top 2-3-3 vertex must be equivalent modulo total derivatives and linearized equations of motion to the nonabelian 2-3-3 vertex presented in appendix B of [7] which we have verified explicitly.[6]

1.5 Uniqueness of the $2-3-3$ FV vertex

The uniqueness of the FV cancelation procedure in the case of spin $s=3$ can be now be established for any D as follows. We obtained the AdS_D covariantization

$$S^\Lambda[h,\phi] = S^\Lambda_{\text{free}} + g\,S^\Lambda_{\text{cubic}}$$

of the nonabelian flat spacetime action

$$S^{\Lambda=0}[h,\phi] = S^{\text{Flat}}_{\text{free}} + g\,S^{\text{Flat}}_{\text{cubic}}$$

obtained in [7], with $S^{\Lambda=0}_{\text{cubic}} = \int d^Dx\, V^{(4)}_0(2,3,3)$ and g the deformation parameter. The cubic part $S^\Lambda_{\text{cubic}} = \int d^Dx\sqrt{-g}\;V_\Lambda(2,3,3)$ possesses an expansion in powers of the AdS radius, where the contribution to $V_\Lambda(2,3,3)$ with the maximum number of derivatives is called $V^{\text{top}}_\Lambda(2,3,3)$. We recall that, using the power of the BRST-BV cohomological method [9], the first-order deformation $S^{\Lambda=0}[h,\phi]$ has been proved [7] to be *unique* under the sole assumptions of

- Locality,
- Manifest Poincaré invariance,
- Nonabelian nature of the deformed gauge algebra.

The last assumption allows the addition of Born-Infeld-like cubic vertices of the form $V_0^{BI}(2,3,3) = C(h)C(\phi)C(\phi)$ where $C(h)$ and $C(\phi)$ denote linearized Weyl tensors and we note that $C(\phi)$ contains 3 derivatives [19]. Such vertices are strictly gauge invariant and do not deform the gauge algebra nor the transformations. We also disregard deformations

[6]Modulo the Bianchi identities of $R_{\mu\nu}$, there are 49 four-derivative terms that are proportional to the spin-2 field equations: 25 terms of the form $R_{..}\partial^2_{..}\phi_{...}\phi_{...}$ and 24 terms of the form $R_{..}\partial_{.}\phi_{...}\partial_{.}\phi_{...}$. Adding an arbitrary linear combination of these to the vertex, lifting the derivatives from $\tilde{h}$, subtracting the "cohomological" vertex, and finally factoring out the spin-3 equations of motion by eliminating $\partial^2\phi$, yields a simple system of equations that allow us to fit the coefficients.

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of the transformations that do not induce nonabelian gauge algebras, as is the case for such deformations involving the curvature tensors. In the following, when we refer to a deformation as unique it should be understood to be up to the addition of other deformations that do not deform the gauge algebra.

The uniqueness of $S^{\Lambda=0}[h,\phi]$ is instrumental in showing the uniqueness of its AdS_D completion $S^{\Lambda}[h,\phi]$, due to the linearity of the perturbative deformation scheme and the smoothness of the flat limit at the level of cubic actions. The proof goes as follows. First suppose that there exists another action $S'^{\Lambda}[h,\phi] = S^{\Lambda}_{\rm free} + g\, S'^{\Lambda}_{\rm cubic}$ that admits a nonabelian gauge algebra and whose top vertex $V'^{\rm top}_{\Lambda}(2,3,3)$ involves $n_{\rm top}$ derivatives with $n_{\rm top}\neq 4$. Then, this action would scale to a nonabelian flat-space action whose cubic vertex would involve $n_{\rm top}$ derivatives. This is impossible, however, because the *only* nonabelian cubic vertex in flat space is $V^{(4)}_0(2,3,3)$. Secondly, suppose there exists a nonabelian action $S''^{\Lambda}[h,\phi] = S^{\Lambda}_{\rm free} + g\, S''^{\Lambda}_{\rm cubic}$ whose top vertex contains 4 derivatives but is otherwise different from $V^{\rm top}_{\Lambda}(2,3,3)$. Then its flat limit would yield a theory with a cubic vertex, involving 4 derivatives, but different from $V^{\rm top}_0(2,3,3)$, which is impossible due to the uniqueness of the latter deformation. Thirdly, and finally, suppose there exists a cubic action with top vertex $V^{\rm top}_{\Lambda}(2,3,3)$ but differing from $S^{\Lambda}_{\rm cubic}$ in the vertices with lesser numbers of derivatives. By the linearity of the BRST-BV deformation scheme, the difference between this coupling and $S^{\Lambda}_{\rm cubic}$ would lead to a nonabelian theory in AdS with top vertex involving less than 4 derivatives. Its flat-space limit would therefore yield a nonabelian action whose top vertex would possess less than 4 derivatives, which is impossible due to the uniqueness of $S^{\Lambda=0}[h,\phi]$.

A more rigorous proof can be stated entirely in terms of master actions within the BRST-BV framework. Then all ambiguities resulting from trivial field and gauge parameter redefinitions are automatically dealt with cohomologically. Moreover, the possibility of scaling away the nonabelianess while at the same time retaining the vertex is ruled out.[7]

1.6 On separation of scales in higher-spin gauge theory

Thanks to Vasiliev's oscillator constructions [20, 21] it has been established that fully nonlinear nonabelian higher-spin gauge field equations exist in arbitrary dimensions in the case of symmetric rank-s tensor gauge fields. Compared to the cubic actions, the full equations exhibit two additional essential features: (i) a precise spectrum $\mathfrak{D}$ given by an *infinite tower* of $\mathfrak{so}(D+1;\mathbb{C})$ representations forming a unitary representation of (a real form of) the higher-spin algebra $\mathfrak{h}$ (see e.g. [22]); (ii) nonlocal, potentially *infinite*, Born-Infeld tails.

The closed form of Vasiliev's equations requires the *unfolded formulation* of field theory whereby [16, 23–25]: (i) standard physical (gauge) fields are replaced as independent

[7]Consider a master action $W_\lambda = \overset{(0)}{W}_\lambda + g\,\overset{(1)}{W}_\lambda + \cdots$ with $\overset{(1)}{W}_\lambda = \int (a_2^\lambda + a_1^\lambda + a_0^\lambda)$ where a_2^λ, a_1^λ and a_0^λ, respectively, contain the nonabelian deformation of the gauge algebra, the corresponding gauge transformations and vertices. The master equation amounts to $\gamma^\lambda a_2^\lambda = 0$, $\gamma^\lambda a_1^\lambda + \delta^\lambda a_2^\lambda = dc_1^\lambda$ and $\gamma^\lambda a_0^\lambda + \delta^\lambda a_1^\lambda = dc_0^\lambda$ where γ^λ and δ^λ have λ expansions starting at order λ^0. Since the system is linear and determines a_1^λ and a_0^λ for given a_2^λ it follows that all a_i^λ scale with λ the same way in the limit $\lambda \to 0$.

action variables by differential forms taking their values in $\mathfrak{so}(D+1;\mathbb{C})$ modules that are finite-dimensional for p-forms with $p>0$ and infinite-dimensional for zero-forms; (ii) the resulting kinetic terms feature only the exterior derivative d; (iii) the standard interactions are mapped to non-linear structure functions appearing in the unfolded first-order equations obeying *algebraic conditions* assuring $d^2=0$. Thus the on-shell content of a spin-s gauge field ϕ_s is mapped into an infinite-dimensional collection of zero-forms carrying traceless Lorentz indices filling out the covariant Taylor expansion on-shell of the corresponding Weyl tensor $C(\phi_s)$. Letting X^α denote the complete unfolded field content, the unfolded equations take the form $dX^\alpha + f^\alpha(X^\beta)=0$ where f^α are written entirely using exterior algebra, and subject to the algebraic condition $f^\beta \frac{\partial^l}{\partial X^\beta} f^\alpha = 0$ (defining what is sometimes referred to as a free-differential algebra [26−28]). The salient feature of the unfolded framework is that any consistent deformation is automatically gauge-invariant in the sense that every p-form with $p\geqslant 1$ is accompanied by a $(p-1)$-form gauge parameter, independently of whether the symmetry is manifestly realized or not.

Vasiliev's equations provide one solution to the on-shell deformation problem given a one-form A taking its values in the algebra $\mathfrak{h}$, and a zero-form Φ containing all Weyl tensors and their on-shell derivatives, which is the unfolded counterpart of the massless representation $\mathfrak{D}$. The embedding of the canonical fields $\{g_{\mu\nu},\phi,\dots\}$ into Φ and A requires a non-local field redefinition[8] to microscopic counterparts $\{\widehat{g}_{\mu\nu},\widehat{\phi},\dots\}$. In the microscopic frame, the standard field equations are non-canonical and actually contain infinite Born-Infeld tails already at first order in the weak-field expansion (see [30] for a discussion). For example, the first-order corrections to the stress tensor, defined by $\widehat{R}_{\mu\nu}-\frac{1}{2}\widehat{g}_{\mu\nu}(\widehat{R}-\Lambda)=\widehat{T}_{\mu\nu}$, from a given spin s arise in a derivative expansion of the form $\widehat{T}^{(1)}_{\mu\nu} = \sum_{n=0}^\infty \sum_{p+q=2n} \lambda^{-2n}\widehat{T}^{(n);p,q}_{\mu\nu}(\widehat{\nabla}^p\widehat{\phi}_s,\widehat{\nabla}^q\widehat{\phi}_s)$ where $\widehat{\nabla}^p\widehat{\phi}_s$ is a connection if $p<s$ and $(p-s)$ derivatives of $C(\widehat{\phi}_s)$ if $p\geqslant s$ (see e.g. [31] for the case of $s=0$).

As discussed below (1.12), the microscopic tails should be related to the canonical vertices via non-local, potentially divergent, field redefinitions. Thus one has the following scheme:

$$\begin{array}{ccccc} \text{Unfolded} & & \text{Standard-exotic} & & \text{Standard-exotic} \\ \text{master-field} & \overset{\text{weak fields}}{\leftrightarrows} & \text{microscopic} & \overset{\text{non-local field redef.}}{\rightleftarrows} & \text{canonical} \\ \text{equations} & & \text{field equations} & & \text{field equations} \end{array} \qquad (1.21)$$

Thus, the weak-field expansion, whether performed in the microscopic or canonical frames, leads to amplitudes depending on the following three quantities: (i) a dimensionless AdS-Planck constant $g^2 \equiv (\lambda\ell_p)^{D-2}$ that can always be taken to obey $g\ll 1$ and that counts the order in the perturbative weak-field expansion, where ℓ_p enters via the normalization of the effective standard action and we are working with dimensionless physical fields; and (ii) a massive parameter λ that simultaneously (iia) sets the infrared cutoff via $\Lambda\sim\lambda^2$ and critical masses $M^2\sim\lambda^2$ for the dynamical fields; and (iib) dresses the derivatives in

[8]The situation in higher-spin gauge theory is analogous to that in string theory: in both cases the microscopic formulation is defined in terms of "vertex operators" living in an associative algebra associated with an "internal" quantum theory. As a result, the graviton vertex receives corrections leading to a microscopic frame that is different from the canonical Einstein frame (see [29] for a related discussion).

the interaction vertices thus enabling the Fradkin-Vasiliev (FV) mechanism; and (iii) the weak-field fluctuation amplitudes[9] $|\nabla^n C(\phi)| \sim (\lambda\ell)^{-s-n}|\phi|$ where ℓ is the characteristic wavelength of the bulk fields.

We stress that what makes higher-spin theory exotic is the dual purpose served by λ within the FV mechanism whereby positive and negative powers of λ appear in critical mass terms and higher-derivative vertices, respectively. Since the critical masses serve as infra-red cutoffs, fluctuations around backgrounds that are close to the AdS_D solution (with unbroken higher-spin symmetry) have derivatives scaling like $(\ell\lambda)^{-1} \gg 1$. Thus, in the canonical weak-field expansion, the connected contributions $\frac{1}{\ell_p^{D-2}}\int V_\Lambda(s_1,\dots,s_N)$ to N-point amplitudes $A(s_1,\dots,s_N|\ell\lambda;g)$ are dominated in the classical limit by strongly coupled top-vertices $V_\Lambda^{(\rm top)}(s_1,\dots,s_N)$ necessarily containing total numbers of derivatives $n_{\rm top}(\{s_i\})$ growing at least linearly with $\sum_i s_i$, suggesting that $A(s_1,\dots,s_N|\ell\lambda;g) \sim g^{N-2}(\ell\lambda)^{-n_{\rm top}(\{s_i\})}$ (in particular, the contribution from the standard minimal gravitational two-derivative couplings are washed out by the contributions from their associated top-vertices). The non-uniform scaling behavior in $\{s_i\}$ seems problematic, in comparison with ordinary (non-exotic) field theories, in the sense that the perturbative expansion cannot be made weakly coupled by choosing g small enough.

On the other hand, in the microscopic weak-field expansion scheme, the corresponding amplitudes $\widehat{A}(s_1,\dots,s_N|\ell\lambda;g)$, computed using the microscopic field variables, contain connected parts given by Born-Infeld tails which are power-series expansions in $z=(\ell\lambda)^{-1}$. These tails define special functions in the unphysical region $|z| \ll 1$ that one might try to continue into the physical region $|z| \gg 1$. The above discussions suggest that the amplitudes should be evaluated directly within Vasiliev's master-field formalism, where the microscopic fields are embedded into master fields that are operators (built perturbatively from the microscopic fields and internal oscillators) that by assumption must belong to an associative algebra in order for the full master-field equations to be consistent and hence gauge invariant. Thus the problem of handling potentially divergent tails while maintaining higher-spin gauge invariance is mapped to the possibly more transparent problem of regularizing operator products while maintaining associativity. Indeed, related regularizations of operator products have been examined and found to be useful in the context of classical solutions [29].

Finally, one may speculate that the non-uniform scaling behavior of the canonical amplitudes $A(s_1,\dots,s_N|\ell\lambda;g)$ might change after regularization such that the analytically continued microscopic amplitudes $\widehat{A}(s_1,\dots,s_N|\ell\lambda;g)$ are bounded uniformly in $\{s_i\}$ for $\ell\lambda \ll 1$, making the perturbative expansion well-defined for $g \ll 1$, at least semi-classically.

2. Antifield formulation

2.1 Definitions

In this section we briefly recall the BRST deformation scheme [9] in the case of spin-s

[9]The gauge- invariant characterization of the amplitudes is provided by on-shell closed forms built from Φ and A. A simple set of such "observables" are the zero-form charges found in [32].

Fronsdal theory, that is irreducible and abelian. The containt of the present section is mainly based on the works [33 − 35].

According to the general rules of the BRST-antifield formalism, a Grassmann-odd ghost is introduced, which accompanies each Grassmann-even gauge parameter of the gauge theory. It possesses the same algebraic symmetries as the corresponding gauge parameter. In the cases at hand, it is symmetric and traceless in its spacetime indices. Then, to each field and ghost of the spectrum, a corresponding antifield (or antighost) is added, with the same algebraic symmetries but the opposite Grassmann parity. A $\mathbb{Z}$-grading called *ghost number* (*gh*) is associated with the BRST differential s, while the *antifield number* (*antigh*) of the antifield Z^* associated with the field (or ghost) Z is given by $antigh(Z^*) \equiv gh(Z)+1$. It is also named *antighost number*. More precisely, in the general class of theories under consideration, the spectrum of fields (including ghosts) and antifields together with their respective ghost and antifield numbers is given by ($s > 2$)

- the fields $\{A_\mu, h_{\mu\nu}, \phi_{\mu_1\ldots\mu_s}\}$ with ghost number 0 and antifield number 0;
- the ghosts $\{C, C_\mu, C_{\mu_1\ldots\mu_{s-1}}\}$ with ghost number 1 and antifield number 0;
- the antifields $\{A^{*\mu}, h^{*\mu\nu}, \phi^{*\mu_1\cdots\mu_s}\}$, with ghost number -1 and antifield number 1;
- the antighosts $\{C^*, C^{*\mu}, C^{*\mu_1\cdots\mu_{s-1}}\}$ with ghost number -2 and antifield number 2 .

If the *pureghost number* (*pgh*) of an expression simply gives the number of ghosts (and derivatives of the ghosts) present in this expression, the ghost number (*gh*) is simply given by

$$gh = pgh - antigh \; .$$

The fields and ghosts will sometimes be denoted collectively by Φ_I , the antifields by Φ^{*I}.

The basic object in the antifield formalism is the BRST generator W_0 . For a spin-1 field A_μ , a spin-2 field $h_{\mu\nu}$ and a (double-traceless) spin-s Fronsdal field $\phi_{\mu_1\ldots\mu_s}$, it reads

$$\begin{aligned}
W_{0,1} &= S_{EM}[A_\mu] + \int A^{*\mu}\, \partial_\mu C \; d^D x \;, \\
W_{0,2} &= S_{PF}[h_{\mu\nu}] + 2 \int h^{*\mu\nu}\, \partial_{(\mu} C_{\nu)} \; d^D x \,, \\
W_{0,s} &= S_F[\phi_{\mu_1\ldots\mu_s}] + s \int \phi^{*\mu_1\ldots\mu_s}\, \partial_{(\mu_1} C_{\mu_2\ldots\mu_s)} \; d^D x \;.
\end{aligned}$$

The functional W_0 satisfies the master equation $(W_0, W_0) = 0$, where $(\, ,)$ is the antibracket given by

$$(A, B) = \frac{\delta^R A}{\delta \Phi_I}\frac{\delta^L B}{\delta \Phi^{*I}} - \frac{\delta^R A}{\delta \Phi^{*I}}\frac{\delta^L B}{\delta \Phi_I} \,. \tag{2.1}$$

Let us note that this definition is appropriate for both functionals and differentials forms. In the former case, the summation over I also implies an integration over spacetime (de Witt's condensed notation). See the textbook [36] for a thorough exposition of the BRST formalism.

The action of the BRST differential s is defined by

$$sA = (W_0\,, A)\ .$$

The differential s is the sum of the Koszul-Tate differential δ (which reproduces the equations of motion and the Noether identities) and the longitudinal derivative γ (which reproduces the gauge transformations and the gauge algebra). Let us write down explicitly the action of δ and γ (unless it is vanishing): For a spin-1 field:

$$\delta C^* = -\partial_\mu A^{*\mu}\,,\quad \delta A^{*\mu} = \partial_\rho F^{\rho\mu}\,,\quad \gamma A_\mu = \partial_\mu C\ .$$

For a spin-2 field:

$$\delta C^{*\nu} = -2\,\partial_\mu h^{*\mu\nu}\,,\quad \delta h^{*\mu\nu} = -2\,H^{\mu\nu}\,,\quad \gamma h_{\mu\nu} = 2\,\partial_{(\mu}C_{\nu)}\ .$$

For a spin-s field:

$$\delta C^{*\mu_1\ldots\mu_{s-1}} = -s\left(\partial_{\mu_s}\phi^{*\mu_1\ldots\mu_s} - \frac{(s-1)(s-2)}{2(D+2s-6)}\,\eta^{(\mu_1\mu_2}\partial_{\mu_s}\phi'^{*\mu_3\ldots\mu_{s-1})\mu_s}\right)\,,$$

$$\delta\phi^{*\mu_1\ldots\mu_s} = G^{\mu_1\ldots\mu_s}\,,\quad \gamma\phi_{\mu_1\ldots\mu_s} = s\,\partial_{(\mu_1}C_{\mu_1\ldots\mu_s)}\,,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $K_{\alpha\beta|\mu\nu} = -\frac{1}{2}(\partial^2_{\alpha\mu}h_{\beta\nu} + \partial^2_{\beta\nu}h_{\alpha\mu} - \partial^2_{\alpha\nu}h_{\beta\mu} - \partial^2_{\beta\mu}h_{\alpha\nu})$ are the Maxwell field-strength and the linearized Riemann tensor, respectively.[10] The linearized Einstein tensor is $H_{\mu\nu} = K_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}K$ where $K_{\beta\nu} = \eta^{\alpha\mu}\,K_{\alpha\beta|\mu\nu}$ is the linearized Ricci tensor and $K = \eta^{\beta\nu}\,K_{\beta\nu}$ the linearized scalar curvature. Finally, the flat-spacetime spin-s Einstein-like and Fronsdal tensors $G_{\mu_1\ldots\mu_s}$ and $F_{\mu_1\ldots\mu_s}$ are given by

$$\begin{aligned} G_{\mu_1\ldots\mu_s} &= F_{\mu_1\ldots\mu_s} - \frac{s(s-1)}{4}\,\eta_{(\mu_1\mu_2}\,F'_{\mu_3\ldots\mu_s)}\,,\\ F_{\mu_1\ldots\mu_s} &= \Box\phi_{\mu_1\ldots\mu_s} - s\,\partial^2_{\rho(\mu_1}\phi^\rho{}_{\mu_2\ldots\mu_s)} + \frac{s(s-1)}{2}\,\partial^2_{(\mu_1\mu_2}\phi'_{\mu_3\ldots\mu_s)}\ . \end{aligned}$$

For further purposes we also display the spin-s curvature

$$K_{\mu_1\nu_1|\ldots|\mu_s\nu_s} = 2^s\,Y^s(\partial^s_{\mu_1\ldots\mu_s}\phi_{\nu_1\ldots\nu_s})\,,\quad s>2\,, \tag{2.2}$$

where we have used the permutation operator

$$Y^s = \frac{1}{2^s}\prod_{i=1}^{s}[e - (\mu_i\nu_i)]$$

that performs total antisymmetrization over the pairs of indices (μ_i,ν_i), $i = 1,\ldots,s$. Finally, we note that the Fronsdal and curvature tensors are not quite independent. The following relations can be established:

$$\begin{aligned} K^\rho{}_{\nu_{s-1}|\rho\nu_s|\mu_1\nu_1|\ldots|\mu_{s-2}\nu_{s-2}} &= 2^{s-2}\,Y^{s-2}(\partial^{s-2}_{\mu_1\ldots\mu_{s-2}}F_{\nu_1\ldots\nu_s})\,,\\ \partial^{\mu_s}K_{\mu_1\nu_1|\ldots|\mu_s\nu_s} &= 2^{s-1}\,Y^{s-1}(\partial^{s-1}_{\mu_1\ldots\mu_{s-1}}F_{\nu_1\ldots\nu_s})\ . \end{aligned}$$

In the following two subsections we give some cohomological results needed for the BRST-BV analysis of the deformation problem.

[10]We use the notation $\partial^N_{\mu_1\ldots\mu_N} \equiv \partial_{\mu_1}\ldots\partial_{\mu_N}$.

2.2 Cohomology $H^*(\gamma)$

For a proof of general results, see [33]. The only gauge-invariant functions for a spin-s gauge field are functions of the field-strength tensor $F_{\mu\nu}$, the Riemann tensor $K_{\alpha\beta|\mu\nu}$, the Fronsdal tensor $F_{\mu_1\ldots\mu_s}$ and the curvature tensor $K_{\mu_1\nu_1|\mu_2\nu_2|\ldots|\mu_s\nu_s}$. In pureghost number $pgh = 0$ one has: $H^0(\gamma) = \{f([F_{\mu\nu}], [K], [F_s], [K_s], [\Phi^{*I}])\}$ where the notation $[\psi]$ indicates the (anti)field ψ as well as all its derivatives up to a finite (but otherwise unspecified) order. In $pgh > 0$, it can be shown (along the same lines as in [7], appendix A) that one can choose $H^*(\gamma)$-representatives as the products of an element of $H^0(\gamma)$ with an appropriate number of non γ-exact ghosts. The latter are $\{\ C,\ C_\mu,\ \partial_{[\mu}C_{\nu]},\ C_{\mu_1\ldots\mu_{s-1}}\ \}$ together with the traceless part of $Y^j(\partial_{\mu_1\ldots\mu_j}C_{\nu_1\ldots\nu_{s-1}})$ for $j \leqslant s-1$, that we denote $U^{(j)}_{\mu_1\nu_1|\ldots\mu_j\nu_j|\nu_{j+1}\ldots\nu_{s-1}}$. If we denote by ω^i_J a basis of the products of these objects in $pgh = i$, we get:

$$H^i(\gamma) \cong \{\alpha^J\omega^i_J \mid \alpha^J \in H^0(\gamma)\}\ . \tag{2.3}$$

More generally, let $\{\omega_I\}$ be a basis of the space of polynomials in these variables (since these variables anticommute, this space is finite-dimensional). If a local form a is γ-closed, we have

$$\gamma a = 0 \quad \Rightarrow \quad a \,=\, \alpha^J\,\omega_J + \gamma b\,. \tag{2.4}$$

If a has a fixed, finite ghost number, then a can only contain a finite number of antifields. Moreover, since the *local* form a possesses a finite number of derivatives, we find that the α^J are polynomials. Such a polynomial α^J will be called an *invariant polynomial*.

We shall need several standard results on the cohomology of d in the space of invariant polynomials.

Proposition 1. *In form degree less than D and in antifield number strictly greater than 0, the cohomology of d is trivial in the space of invariant polynomials. That is to say, if α is an invariant polynomial, the equation $d\alpha = 0$ with $antigh(\alpha) > 0$ implies $\alpha = d\beta$ where β is also an invariant polynomial.*

The latter property is rather generic for gauge theories (see e.g. ref. [34] for a proof), as well as the following:

Proposition 2. *If a has strictly positive antifield number, then the equation $\gamma a + db = 0$ is equivalent, up to trivial redefinitions, to $\gamma a = 0$. More precisely, one can always add d-exact terms to a and get a cocycle $a' := a + dc$ of γ, such that $\gamma a' = 0$.*

2.3 Homological groups $H^D_2(\delta|d)$ and $H^D_2(\delta|d, H^0(\gamma))$

We first recall a general result (theorem 9.1 in [37]):

Proposition 3. *For a linear gauge theory of reducibility order r,*

$$H^D_p(\delta|\, d) = 0\ for\ p > r+2\,.$$

Since the theory at hand has no reducibility, we are left with the computation of $H_2^D(\delta|\,d)$. Then, as we already claimed in [7], for a collection of different spins, $H_2^D(\delta|d)$ is the direct sum of the homologies of the individual cases.

For spin-1:

$$H_2^D(\delta|d) = \left\{\Lambda C^*\, d^D x \mid \Lambda \in \mathbb{R}\right\}.$$

For spin-2 :

$$H_2^D(\delta|d) = \left\{\xi^\mu C^*_\mu\, d^D x \mid \partial_{(\mu}\xi_{\nu)} = 0\right\}. \tag{2.5}$$

For spin-s ($s > 2$) ([33, 38]):

$$H_2^D(\delta|d) = \left\{\xi^{\mu_1\ldots\mu_{s-1}} C^*_{\mu_1\ldots\mu_{s-1}}\, d^D x \mid \partial_{(\mu_1}\xi_{\mu_2\ldots\mu_s)} = 0\right\}. \tag{2.6}$$

2.4 BRST deformation

As shown in [9], the Noether procedure can be reformulated within a BRST-cohomological framework. Any consistent deformation of the gauge theory corresponds to a solution

$$W = W_0 + gW_1 + g^2 W_2 + \mathscr{O}(g^3)$$

of the deformed master equation $(W, W) = 0$. Taking into account field-redefinitions, the first-order nontrivial consistent local deformations $W_1 = \int a^{D,0}$ are in one-to-one correspondence with elements of the cohomology $H^{D,0}(s|\,d)$ of the zeroth order BRST differential $s = (W_0\,, \cdot)$ modulo the total derivative d, in maximum form-degree D and in ghost number 0. That is, one must compute the general solution of the cocycle condition

$$sa^{D,0} + db^{D-1,1} = 0\,, \tag{2.7}$$

where $a^{D,0}$ is a top-form of ghost number zero and $b^{D-1,1}$ a $(D-1)$-form of ghost number one, with the understanding that two solutions of (2.7) that differ by a trivial solution should be identified

$$a^{D,0} \sim a^{D,0} + sp^{D,-1} + dq^{D-1,0}$$

as they define the same interactions up to field redefinitions. The cocycles and coboundaries $a, b, p, q, \ldots$ are local forms of the field variables (including ghosts and antifields). The corresponding second-order interactions W_2 must satisfy the consistency condition

$$sW_2 = -\frac{1}{2}(W_1, W_1)\,.$$

This condition is controlled by the local BRST cohomology group $H^{D,1}(s|d)$.

Quite generally, one can expand a according to the antifield number, as

$$a = a_0 + a_1 + a_2 + \ldots a_k\,, \tag{2.8}$$

where a_i has antifield number i. The expansion stops at some finite value of the antifield number by locality, as was proved in [39].

Let us recall [10] the meaning of the various components of a in this expansion. The antifield-independent piece a_0 is the deformation of the Lagrangian; a_1, which is linear in the antifields associated with the gauge fields, contains the information about the deformation of the gauge symmetries; a_2 contains the information about the deformation of the gauge algebra (the term C^*CC gives the deformation of the structure functions appearing in the commutator of two gauge transformations, while the term $\phi^*\phi^*CC$ gives the on-shell closure terms); and the a_k ($k > 2$) give the informations about the deformation of the higher order structure functions and the reducibility conditions.

In fact, using standard reasonings (see e.g. [34]), one can remove all components of a with antifield number greater than 2. The key point, as explained e.g. in [35], is that the invariant characteristic cohomology $H_k^{n,inv}(\delta|d)$ controls the obstructions to the removal of the term a_k from a and that all $H_k^{n,inv}(\delta|d)$ vanish for $k > 2$ by Proposition 3 and theorem 2 proved in section B. This proves the first part of the following theorem 1, valid up to spin $s = 4$:

Theorem 1. *Let a be a local top form which is a nontrivial solution of the equation (2.7). Without loss of generality, one can assume that the decomposition (2.8) stops at antighost number two, i.e.*

$$a = a_0 + a_1 + a_2 \,. \tag{2.9}$$

Moreover, the element a_2 is cubic: linear in the antighosts and quadratic in the variables $\{C, C_\mu, \partial_{[\mu}C_{\nu]}, C_{\mu_1\ldots\mu_{s-1}}, U^{(j\leqslant s-1)}_{\mu_1\nu_1|\ldots|\mu_j\nu_j|\nu_{j+1}\ldots\nu_{s-1}}\}|_{s\leqslant 4}$ given in subsection 2.2.

Similarly to (2.9), one can assume $b = b_0 + b_1$ in (2.7) (see e.g. [34]) and insert the expansions of a and b into the latter equation. Decomposing the BRST differential as $s = \delta + \gamma$ yields

$$\gamma a_0 + \delta a_1 + db_0 = 0 \,, \tag{2.10}$$

$$\gamma a_1 + \delta a_2 + db_1 = 0 \,, \tag{2.11}$$

$$\gamma a_2 = 0 \,. \tag{2.12}$$

The general solution of (2.12) is given in subsection 2.2.

Remark. Actually, even if the theorem 2 cannot be extended to $s > 4$ for technical reasons, we can always assume that a_2 is cubic as given in the above theorem 1, relax the limitation $s \leqslant 4$ and proceed with the determination of a_1 and a_0 according to (2.11) and (2.10). In fact, it is impossible to build a ghost-zero cubic object with *antigh* > 2, so a cubic deformation always stops at *antigh* 2. Moreover, a cubic element a_2 must be proportional to an antighost and quadratic in the ghosts, then, modulo d and γ, it is obvious that the only possible cubic deformations are those given in theorem 1. Finally, combining the cohomological approach with other approaches like the light-cone one [4, 5] may complete our results, as we actually show in the following. Such a combination of two different methods seems to us the most powerful way to completely solve the first-order deformation problem.

3. Consistent vertices $V^{\Lambda=0}(1, s, s)$

In this section we use the antifield formalism reviewed above and apply it to the study of nonabelian interactions between spin-1 and spin-s gauge fields. We first examine in detail the interactions of the type $1-2-2$, and then move on to the general case $1-s-s$.

3.1 Exotic nonabelian vertex $V^{\Lambda=0}(1,2,2)$

In this section we show the existence of a cubic cross-interaction between a spin 1 field and a family of exotic spin 2 fields. The structure constants of this vertex are antisymmetric, which is in contradiction with the result for self-interacting spin 2 fields (see [34]). In fact, we easily prove that this vertex cannot coexist with the Einstein- Hilbert theory.

We consider in the following a set of fields in Minkowski spacetime of dimension D. First, a single electromagnetic field A_μ with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, invariant under the gauge transformations $\overset{(0)}{\delta}_\Lambda A_\mu = \partial_\mu \Lambda$. Then, a family of Fierz-Pauli fields $h^a_{\mu\nu}$ where a is the family index. The linearized Riemann tensor is $K^a_{\alpha\beta|\mu\nu} = -\frac{1}{2}(\partial^2_{\alpha\mu}h^a_{\beta\nu} + \partial^2_{\beta\nu}h^a_{\alpha\mu} - \partial^2_{\alpha\nu}h^a_{\beta\mu} - \partial^2_{\beta\mu}h^a_{\alpha\nu})$. It is invariant under the linearized diffeomorphism gauge transformations $\overset{(0)}{\delta}_\xi h^a_{\mu\nu} = 2\partial_{(\mu}\xi^a_{\nu)}$. The linearized Einstein tensor is $H^a_{\mu\nu} = K^a_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}K^a$, according to the notation given in section 2.1.

The free action is the sum of the electromagnetic action and the different Pauli-Fierz actions:

$$S_0 = \int \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - h^{\mu\nu}_a H^a_{\mu\nu} \right) d^D x \,. \tag{3.1}$$

In order to study the cubic deformation problem efficiently, we have used the antifield formalism of [9], reviewed in the present paper. The antifield formalism allows us to write down every possible nontrivial deformation of the gauge algebra, encoded in the element denoted a_2 above. It turns out that only one a_2 candidate gives rise to a consistent vertex a_0. The details of the analysis are relegated to the appendix A, not to obscure the reading. The cubic vertex, gauge transformations and gauge algebra are

$$\begin{aligned}
\overset{(3)}{\mathscr{L}} &= l_{[ab]} \left[-F^{\rho\sigma}\partial_{[\mu}h^a_{\nu]\rho}\partial^\mu h^{b\nu}{}_\sigma + 2A^\sigma K^a_{\mu\nu|\rho\sigma}\partial^\mu h^{b\nu\rho} \right] , \\
\overset{(1)}{\delta}_\xi A_\rho &= 2l_{[ab]}\partial_{[\mu}h^a_{\nu]\rho}\partial^\mu \xi^{b\nu} \\
\overset{(1)}{\delta}_{\xi,\Lambda} h_{a\nu\rho} &= 2l_{[ab]}\left[\Lambda K^b_{\nu\rho} + \frac{1}{2}F^\mu{}_\rho \partial_{[\mu}\xi^b_{\nu]} + \frac{1}{2}F^\mu{}_\nu \partial_{[\mu}\xi^b_{\rho]}\right] - \frac{1}{D-2} l_{[ab]}\eta_{\nu\rho}\left[\Lambda K^b + F^{\mu\nu}\partial_\mu \xi^b_\nu\right] \\
\left[\overset{(0)}{\delta}_\xi, \overset{(1)}{\delta}_\eta\right] A_\mu &+ \left[\overset{(1)}{\delta}_\xi, \overset{(0)}{\delta}_\eta\right] A_\mu = \partial_\mu \Lambda \quad \text{where} \quad \Lambda = 2l_{[ab]}\partial_{[\mu}\xi^a_{\nu]}\partial^\mu \eta^{b\nu} \,.
\end{aligned} \tag{3.2}$$

3.2 Exotic nonabelian vertex $V^{\Lambda=0}(1,s,s)$

The structure that we have found can be easily extended to obtain a set of consistent $1-s-s$ vertices. Using the notation introduced in section 2.1, the Fronsdal action reads

$$S_{\mathrm{Fs}} = \frac{1}{2} \int \phi^{\mu_1\ldots\mu_s}_a \, G^a_{\mu_1\ldots\mu_s} \, d^D x \,. \tag{3.3}$$

It is gauge invariant thanks to the Noether identities

$$\partial^{\mu_s}G^a_{\mu_1\ldots\mu_s}-\frac{(s-1)(s-2)}{2(D+2s-6)}\eta_{(\mu_1\mu_2}\partial^{\mu_s}G'^a_{\mu_3\ldots\mu_{s-1})\mu_s}\equiv 0$$

and the symmetry of the second-order differential operator defining G.

The deformation analysis is performed exactly along the same lines as for the $1-2-2$ vertex. The uniqueness of the solution has not been proved for spin $s>4$, but we show that it is the only cubic solution deforming the gauge algebra. The spin-2 solution can be extended to spin s, which leads us to consider a deformation of the BRST generator stoping at antighost 2, finishing with the following a_2:

$$a_2=f_{[ab]}C^*Y^{s-1}(\partial^{s-1}_{\mu_1\ldots\mu_{s-1}}C^a_{\nu_1\ldots\nu_{s-1}})Y^{s-1}(\partial^{s-1\mu_1\ldots\mu_{s-1}}C^{b\nu_1\ldots\nu_{s-1}})d^Dx\,. \tag{3.4}$$

By solving the equation $\delta a_2+\gamma a_1=db_1$, we first obtain

$$a_1=\tilde{a}_1+\bar{a}_1=2f_{[ab]}A^{*\rho}Y^{s-1}(\partial^{s-1}_{\mu_1\ldots\mu_{s-1}}\phi^a_{\nu_1\ldots\nu_{s-1}\rho})Y^{s-1}(\partial^{s-1\mu_1\ldots\mu_{s-1}}C^{b\nu_1\ldots\nu_{s-1}})d^Dx+\bar{a}_1$$

with $\bar{a}_1\mid\gamma\bar{a}_1=de_1$. The resolution of $\delta a_1+\gamma a_0=db_0$ provides us with both $\bar{a}_1$ and a_0:

$$\begin{aligned}\bar{a}_1\;=\;&2f_{[ab]}\partial^{(s-1)\mu_2\ldots\mu_{s-1}}\phi^{*a\rho_1\rho_2\nu_3\ldots\nu_{s-1}\tau}D^{\nu_1\nu_2\sigma}_{\rho_1\rho_2\tau}\times\\&\left[F^{\mu_1}{}_{\sigma}Y^{s-1}(\partial^{s-1}_{\mu_1\ldots\mu_{s-1}}C^b_{\nu_1\ldots\nu_{s-1}})-\frac{1}{2^{s-1}}CK^{b\mu_1}{}_{\sigma|\mu_1\nu_1|\ldots\mu_{s-1}\nu_{s-1}}\right]d^Dx\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}D^{\nu_1\nu_2\sigma}_{\rho_1\rho_2\tau}\;&=\;\delta^{\nu_1}_{\rho_1}\delta^{\nu_2}_{\rho_2}\delta^{\sigma}_{\tau}-\frac{1}{2(D+2s-6)}\eta_{\rho_1\rho_2}\eta^{\sigma\nu_1}\delta^{\nu_2}_{\tau}-\frac{s-2}{D+2s-6}\eta_{\rho_1\rho_2}\eta^{\sigma\nu_2}\delta^{\nu_1}_{\tau}\,,\\a_0\;&=\;-f_{[ab]}F^{\rho\sigma}Y^{s-1}(\partial^{s-1}_{\mu_1\ldots\mu_{s-1}}\phi^a_{\nu_1\ldots\nu_{s-1}\rho})Y^{s-1}(\partial^{s-1\mu_1\ldots\mu_{s-1}}\phi^{b\nu_1\ldots\nu_{s-1}}{}_{\sigma})d^Dx\\&\quad+f_{[ab]}\frac{1}{2^{s-2}}A^\rho K^a_{\mu_1\nu_1|\ldots|\mu_{s-1}\nu_{s-1}|\rho\sigma}Y^{s-1}(\partial^{s-1\mu_1\ldots\mu_{s-1}}\phi^{b\nu_1\ldots\nu_{s-1}\sigma})d^Dx\,.\end{aligned} \tag{3.6}$$

These components of W_1 provide the cubic vertex, the gauge transformations and the gauge algebra:

$$\begin{aligned}\overset{(3)}{\mathscr{L}}\;&=-f_{[ab]}F^{\rho\sigma}Y^{s-1}(\partial^{s-1}_{\mu_1\ldots\mu_{s-1}}\phi^a_{\nu_1\ldots\nu_{s-1}\rho})Y^{s-1}(\partial^{s-1\mu_1\ldots\mu_{s-1}}\phi^{b\nu_1\ldots\nu_{s-1}}{}_{\sigma})\\&\quad+f_{[ab]}\frac{1}{2^{s-2}}\,A^\rho K^a_{\mu_1\nu_1|\ldots|\mu_{s-1}\nu_{s-1}|\rho\sigma}Y^{s-1}(\partial^{s-1\mu_1\ldots\mu_{s-1}}\phi^{b\nu_1\ldots\nu_{s-1}\sigma})\,,\end{aligned} \tag{3.7}$$

$$\overset{(1)}{\delta}_{\xi}\,A_\mu=Y^{s-1}(\partial^{s-1}_{\mu_1\ldots\mu_{s-1}}\phi^a_{\nu_1\ldots\nu_{s-1}\rho})Y^{s-1}(\partial^{s-1\mu_1\ldots\mu_{s-1}}\xi^{b\nu_1\ldots\nu_{s-1}}) \tag{3.8}$$

$$\begin{aligned}\overset{(1)}{\delta}_{\Lambda,\xi}\,\phi_{a\rho_1\rho_2\nu_3\ldots\nu_{s-1}\tau}=&\,2(-1)^{s-1}f_{[ab]}D^{\nu_1\nu_2\sigma}_{\rho_1\rho_2\tau}\partial^{s-1\mu_2\ldots\mu_{s-1}}\Big[F^{\mu_1}{}_{\sigma}Y^{s-1}(\partial^{s-1}_{\mu_1..\mu_{s-1}}\xi^b_{\nu_1\ldots\nu_{s-1}})\\&-\frac{1}{2^{s-1}}\Lambda K^{b\mu_1}{}_{\sigma|\mu_1\nu_1|\ldots|\mu_{s-1}\nu_{s-1}}\Big]\end{aligned} \tag{3.9}$$

$$\left[\overset{(0)}{\delta}_{\xi},\overset{(1)}{\delta}_{\eta}\right]A_\mu+\left[\overset{(1)}{\delta}_{\xi},\overset{(0)}{\delta}_{\eta}\right]A_\mu=\partial_\mu\Lambda \tag{3.10}$$

where

$$\Lambda=2f_{[ab]}Y^{s-1}(\partial^{s-1}_{\mu_1\ldots\mu_{s-1}}\xi^a_{\nu_1\ldots\nu_{s-1}})Y^{s-1}(\partial^{s-1\mu_1\ldots\mu_{s-1}}\eta^{b\nu_1\ldots\nu_{s-1}})\,, \tag{3.11}$$

the other commutators vanishing.

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3.3 Exhaustive list of interactions $V^{\Lambda=0}(1,s,s)$

The uniqueness of the above cubic nonabelian interactions can be obtained by combining the above results with those obtained in [4, 5] using a powerful light-cone method. We learn from the work [4] that there exist only *two* possible cubic couplings between one spin-1 and two spin-s fields. The first coupling involves $2s-1$ derivatives in the cubic vertex whereas the other involves $2s+1$ derivatives. Therefore we conclude that the first coupling corresponds to the nonabelian deformation obtained in the previous subsection. The other one simply is the Born-Infeld-like coupling

$$\overset{(3)}{\mathscr{L}} = g_{[ab]}\, F^{\rho\sigma}\, \eta^{\lambda\tau}\, K^{a}_{\mu_1\nu_1|\ldots|\mu_{s-1}\nu_{s-1}|\rho\lambda}\, K^{b}_{\sigma\tau|}{}^{\mu_1\nu_1|\ldots|\mu_{s-1}\nu_{s-1}}\,, \tag{3.12}$$

which is strictly invariant under the abelian gauge transformations.

4. Uniqueness of the nonabelian $V^{\Lambda=0}(2,4,4)$ vertex

The computation of all the possible nonabelian $2-4-4$ or $4-2-2$ cubic vertices in Minkowski spacetime of arbitrary dimension $D>3$ can be achieved along the same lines as for the $1-s-s$ vertex. We can apply theorem 1 to find a complete list of the possible a_2 terms, thanks to the technical result about $H^{D,inv}_k(\delta|d)$ that we provide in appendix B. Then by solving equations (2.11) and (2.10), we find an unique cubic deformation.

First, it is easily seen that it is impossible to build a non trivial a_2 involving one spin 4 and two spin 2. Then, in the $2-4-4$ case, the highest number of derivatives allowed for a_2 to be nontrivial is 6, but Poincaré invariance imposes an odd number of derivatives. Here is the only a_2 containing 5 derivatives, which gives rise to a consistent cubic vertex:

$$a_2 = f_{AB} C^*_\gamma U^A_{\alpha\mu|\beta\nu|\rho} V^{B\alpha\mu|\beta\nu|\gamma\rho} d^D x$$

where $U^A_{\mu_1\nu_1|\mu_2\nu_2\tau} = Y^2(\partial^2_{\mu_1\mu_2} C^A_{\nu_1\nu_2\tau})$ and $V^A_{\mu_1\nu_1|\mu_2\nu_2|\mu_3\nu_3} = Y^3(\partial^2_{\mu_1\mu_2\mu_3} C^A_{\nu_1\nu_2\nu_3})$.
Then, the inhomogeneous solution of $\delta a_2 + \gamma a_1 = db_1$ can be computed. The structure constants have to be symmetric in order for a_1 to exist: $f_{AB} = f_{(AB)}$

$$a_1 = \tilde{a}_1 + \bar{a}_1 = f_{(AB)} \left[h^*{}_\gamma{}^\sigma \partial^2_{\alpha\beta} \phi^A_{\mu\nu\rho\sigma} V^{B\alpha\mu|\beta\nu|\gamma\rho} - 2h^{*\gamma\sigma} \partial^3_{\alpha\beta[\gamma} \phi^A_{\rho]\mu\nu\sigma} U^{B\alpha\mu|\beta\nu|\rho} \right] d^D x + \bar{a}_1\,.$$

Finally, the last equation is $\delta a_0 + \gamma a_1 = db_0$. It allows a solution, unique up to redefinitions of the fields and trivial gauge transformations. We have to say that the natural writing of a_2 and the vertex written in terms of the Weyl tensor $w_{\alpha\beta|\gamma\delta}$ do not match automatically. In order to get a solution, we first classified the terms of the form $w\partial^4(\phi\phi)$. Then we classified the possible terms in $\bar{a}_1$, which can be chosen in $H^1(\gamma)$. So they are proportional to the field antifields, proportional to a gauge invariant tensor (K_{PF}, F_4 or K_4) and proportional to a non exact ghost. Finally, we had to introduce an arbitrary trivial combination in order for the expressions to match. The computation cannot be made by hand (there are thousands of terms). By using the software FORM [40], we managed to solve the heavy system of equations and found a consistent set of coefficients. We obtained the following $\bar{a}_1$:

$$\bar{a}_1 = \frac{4}{D+2} f_{AB} \phi'^{*A\beta}{}_\alpha \partial^\tau K^{\mu\nu|\alpha\sigma} U^B_{\mu\nu|\beta\sigma|\tau} d^D x - 2 f_{AB} \phi^{*A\mu\rho}{}_{\alpha\beta} \partial^\tau K^{\alpha\nu|\beta\sigma} U^B_{\mu\nu|\rho\sigma|\tau} d^D x\,.$$

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and the cubic vertex:

$$
\begin{aligned}
a_{0,w} \approx f_{AB} w_{\mu\nu\rho\sigma} \Big[& \frac{1}{2}\, \partial^{\mu\rho\alpha} \phi'^{A\nu\beta} \partial_\alpha \phi'^{B\sigma}{}_{\beta} - \frac{1}{3}\, \partial^{\mu\rho\alpha} \phi^{A\nu\beta\gamma\delta} \partial_\alpha \phi^{B\sigma}{}_{\beta\gamma\delta} + \frac{1}{4}\, \partial^{\mu\rho\alpha} \phi^{A\nu\beta\gamma\delta} \partial_\beta \phi^{B\sigma}{}_{\alpha\gamma\delta} \\
& + \frac{3}{4}\, \partial^{\mu\alpha\beta} \phi'^{A\nu\rho} \partial_\alpha \phi'^{B\sigma}{}_{\beta} + \frac{3}{4}\, \partial^{\mu\alpha\beta} \phi^{A\nu\rho\gamma\delta} \partial_\alpha \phi^{B\sigma}{}_{\beta\gamma\delta} - \frac{3}{2}\, \partial^{\mu\alpha\beta} \phi^{A\nu\rho}{}_{\beta\gamma} \partial_\alpha \phi'^{B\sigma\gamma} \\
& - \frac{1}{2}\, \partial^{\mu}{}_{\beta\gamma} \phi^{A\nu\rho\alpha}{}_{\delta} \partial_\alpha \phi^{B\sigma\beta\gamma\delta} - \frac{3}{4}\, \partial^{\mu\alpha\beta} \phi^{A\sigma}{}_{\beta\gamma\delta} \partial_\alpha \phi^{B\nu\rho\gamma\delta} + \frac{3}{2}\, \partial^{\mu\alpha\beta} \phi'^{A\sigma\gamma} \partial_\alpha \phi^{B\nu\rho}{}_{\beta\gamma} \\
& - \partial^{\mu}{}_{\beta\gamma} \phi'^{A\sigma\alpha} \partial_\alpha \phi^{B\nu\rho\beta\gamma} + \frac{1}{2}\, \partial^{\mu}{}_{\beta\gamma} \phi^{A\sigma\alpha\gamma}{}_{\delta} \partial_\alpha \phi^{B\nu\rho\beta\delta} - \frac{1}{2}\, \partial_{\alpha\beta\gamma} \phi^{A\mu\rho\alpha\delta} \partial_\delta \phi^{B\nu\sigma\beta\gamma} \\
& + \frac{1}{2}\, \partial_{\alpha\beta\gamma} \phi^{A\mu\rho\alpha\tau} \partial^\beta \phi^{B\nu\sigma\gamma}{}_{\tau} + \frac{1}{8}\, \partial^{\alpha\beta} \phi'^{A\mu\rho} \partial_{\alpha\beta} \phi'^{B\nu\sigma} + \frac{3}{8}\, \partial^{\alpha\beta} \phi^{A\mu\rho\gamma\delta} \partial_{\alpha\beta} \phi^{B\nu\sigma}{}_{\gamma\delta} \\
& - \frac{1}{2}\, \partial^{\alpha}{}_{\beta} \phi^{A\mu\rho\beta\gamma} \partial_{\alpha\gamma} \phi'^{B\nu\sigma} + \frac{1}{2}\, \partial_{\alpha\beta} \phi^{A\mu\rho\beta\tau} \partial^{\alpha\gamma} \phi^{B\nu\sigma}{}_{\gamma\tau} - \frac{3}{4}\, \partial^{\alpha\beta} \phi^{A\mu\rho\gamma\delta} \partial_{\alpha\gamma} \phi^{B\nu\sigma}{}_{\beta\delta} \\
& + \frac{1}{4}\, \partial_{\alpha\beta} \phi^{A\mu\rho\gamma\delta} \partial_{\gamma\delta} \phi^{B\nu\sigma\alpha\beta} \Big] ,
\end{aligned} \tag{4.1}
$$

where the weak equality means that we omitted terms that are proportional to the free field equations, since they can trivially be absorbed by field redefinitions. The components a_1 and a_2 correspond to the following deformation of the gauge transformations

$$
\begin{aligned}
\overset{(1)}{\delta}_\xi h_{\sigma\tau} &= \frac{1}{2} f_{AB} \Big[\eta_{\tau\mu_3} \partial^2_{\mu_1\mu_2} \phi^A_{\nu_1\nu_2\nu_3\sigma} Y^3 (\partial^{3\mu_1\mu_2\mu_3} \xi^{B\nu_1\nu_2\nu_3}) \\
&\qquad -2 \partial^3_{\mu_1\mu_2[\tau} \phi^A_{\rho]\nu_1\nu_2\sigma} Y^2 (\partial^{2\mu_1\mu_2} \xi^{B\nu_1\nu_2\rho}) \Big] + (\sigma \leftrightarrow \tau)
\end{aligned} \tag{4.2}
$$

$$
\begin{aligned}
\overset{(1)}{\delta}_\xi \phi_{\alpha_1\alpha_2\alpha_3\alpha_4} &= \frac{4}{D+2} f_{AB} \eta_{(\alpha_3\alpha_4} \delta^{\mu_2}_{\alpha_1} \partial^\tau K^{\mu_1\nu_1|}{}_{\alpha_2)}{}^{\nu_2} Y^2 (\partial^2_{\mu_1\mu_2} \xi^B_{\nu_1\nu_2\tau}) \\
&\qquad -2 f_{AB} \delta^{\mu_1}_{(\alpha_1} \delta^{\mu_2}_{\alpha_2} \partial^\tau K_{\alpha_3}{}^{\nu_1}{}_{|\alpha_4)}{}^{\nu_2} Y^2 (\partial^2_{\mu_1\mu_2} \xi^B_{\nu_1\nu_2\tau}) \, .
\end{aligned} \tag{4.3}
$$

and to the following deformation of the gauge algebra:

$$
\Big[\overset{(0)}{\delta}_\xi, \overset{(1)}{\delta}_\eta \Big] h_{\mu\nu} + \Big[\overset{(1)}{\delta}_\xi, \overset{(0)}{\delta}_\eta \Big] h_{\mu\nu} = 2 \partial_{(\mu} j_{\nu)}
$$

where

$$
j_{\mu_3} = f_{(AB)} \partial^{2\mu_1\mu_2} \xi^{A\nu_1\nu_2\nu_3} Y^3 (\partial^3_{\mu_1\mu_2\mu_3} \eta^B_{\nu_1\nu_2\nu_3}) - (\xi \leftrightarrow \eta)
$$

Let us now consider the other possible cases for a_2, containing 3 or 1 derivatives. The only possibility with three derivatives is $a_{2,3} = g_{AB} C^*_\beta \partial_{[\alpha} C^A_{\mu]\nu\rho} U^{B\alpha\mu|\beta\nu|\rho} d^D x$. Its variation under δ should be γ-closed modulo d but some nontrivial terms remain, of the types $g_{AB} h^* U^A U^B$ and $g_{AB} h^* \partial_{[.} C^A_{.]..} V^B$. The first one can be set to zero by imposing symmetric structure constants, but the second cannot be eliminated. The same occurs for one of the candidates with 1 derivative: $a_{2,1,1} = k_{AB} C^{*\beta} C^{A\mu\nu\rho} \partial_{[\beta} C^B_{\mu]\nu\rho} d^D x$. We are then left with 2 candidates involving the spin 4 antifield. We have found that $\delta a_2 + \gamma a_1 = d b_1$ can have a solution only if their structure constants are proportional:

$$
\begin{aligned}
a_{2,1} &= l_{AB} C^{*A\mu\nu\rho} \Big[C^\alpha \partial_{[\alpha} C^B_{\mu]\nu\rho} + 2 \partial_{[\mu} C_{\alpha]} C^{B\alpha}{}_{\nu\rho} \Big] d^D x \\
a_{1,1} &= l_{AB} \phi^{*\mu\nu\rho\sigma} \Big[2 h_\sigma{}^\alpha \partial_{[\alpha} C^B_{\mu]\nu\rho} - \frac{4}{3}\, C^\alpha \partial_{[\alpha} \phi^B_{\mu]\nu\rho\sigma} + 8 \partial_{[\mu} h_{\alpha]\sigma} C^{B\alpha}{}_{\nu\rho} - 2 \partial_{[\mu} C_{\alpha]} \phi^{B\alpha}{}_{\nu\rho\sigma} \Big] \\
&\quad + \frac{2}{D+2} l_{AB} \partial_\sigma \phi'^{*A\rho\sigma} C^\alpha \phi'^B_{\alpha\rho} \, .
\end{aligned}
$$

There is no homogenous part $\bar{a}_1$ (because the γ-invariant tensors contain at least 2 derivatives). Then, we have considered the most general expression for a_0, which is a linear combination of 55 terms of the types $h\phi\partial^2\phi$ and $h\partial\phi\partial\phi$. We have found that the equation $\delta a_1 + \gamma a_0 = db_0$ does not admit any solution. We can conclude that the vertex found with 6 derivatives is the *unique* nonabelian $2-4-4$ cubic deformation.

This $2-4-4$ vertex, setting $D = 4$, should correspond to the flat limit of the corresponding Fradkin-Vasiliev vertex. The uniqueness of the former can be used to prove the uniqueness of the latter, as we did explicitly in the $2-3-3$ case.

5. Consistent vertices $V^{\Lambda=0}(2, s, s)$

5.1 Nonabelian coupling with $2s-2$ derivatives

Our classification of gauge algebra deformations relies on the theorem concerning the homology $H^{D,inv}_k(\delta|d)$. While apparently obviously true, it actually becomes increasingly harder to prove with increasing spin. If $H^{D,inv}_k(\delta|d) = H^D_k(\delta|d) \cap H^0(\gamma)$ holds for spin $s > 4$ then there is only one candidates for a nonabelian type $2-s-s$ deformation, involving $2s-3$ derivatives in a_2.

Let us recall that due to the simple expression of $H^D_2(\delta|d)$ we are only left with a few traceless building blocks for a_2: the antighosts $C^{*\mu}$ and $C^{*A\mu_1\dots\mu_{s-1}}$, and a collection of ghosts and their anti-symmetrized derivatives, namely C_μ, $\partial_{[\mu}C_{\nu]}$ and tensors $U^{(j)A}_{\mu_1\nu_1|\dots|\mu_j\nu_j|\nu_{j+1}\dots\nu_{s-1}}$ for $j \leq s-1$, that we have defined in section 2.2 [33]. Given this, we can divide the a_2 candidates into two categories: those proportional to $C^{*A\mu_1\dots\mu_{s-1}}$ and those proportional to $C^{*\mu}$.

The first category is simple to study: $C^{*A\mu_1\dots\mu_{s-1}}$ carries $s-1$ indices, and the spin 2 ghost can carry at most 2, namely $\partial_{[\alpha}C_{\beta]}$. As no traces can be made, the spin 4 ghost can carry at most $s+1$ indices. But $U^{(2)A}_{\mu_1\nu 1|\mu_2\nu_2|\nu_3\dots\nu_{s-1}}$ contains two antisymmetric pairs which cannot be contracted with C^{*A}. The only possible combination involving $\partial_{[\alpha}C_{\beta]}$ is thus

$$f_{AB}C^{*A\mu_1\dots\mu_{s-1}}\partial_{[\mu_1}C_{\alpha]}C^{B\alpha}{}_{\mu_2\dots\mu_{s-1}}d^Dx \ .$$

If we consider the undifferentiated C_α, the only possibility is obviously:

$$g_{AB}C^{*A\mu_1\dots\mu_{s-1}}C^\alpha U^{(1)A}_{\alpha\mu_1|\mu_2\dots\mu_{s-1}}d^Dx \ .$$

Those two terms contain only one derivative. Just as for the spin 4 case, we can show that they are related to an a_1 if $f_{AB} = \frac{s}{2}g_{AB}$:

$$a_{2,1} = g_{AB}C^{*A\mu_1\dots\mu_{s-1}}\left[C^\alpha\partial_{[\alpha}C^B_{\mu_1]\mu_2\dots\mu_{s-1}} + \frac{s}{2}\partial_{[\mu_1}C_{\alpha]}C^{B\alpha}{}_{\mu_2\dots\mu_{s-1}}\right]d^nx \tag{5.1}$$

and

$$\begin{aligned}a_{1,1} = l_{AB}\phi^{*A\mu_1\dots\mu_s}&\left[\frac{s}{2}h_{\mu_s}{}^\alpha\partial_{[\alpha}C^B_{\mu_1]\mu_2\dots\mu_{s-1}} - \frac{s}{s-1}C^\alpha\partial_{[\alpha}\phi^B_{\mu_1]\mu_2\dots\mu_s}\right.\\ &\left.+\frac{s^2}{2}\partial_{[\mu_1}h_{\alpha]\mu_s}C^{B\alpha}{}_{\mu_2\dots\mu_s} - \frac{s}{2}\partial_{[\mu_1}C_{\alpha]}\phi^{B\alpha}{}_{\mu_2\dots\mu_s}\right]\\ +\frac{s(s-2)}{4(n+2s-6)}&l_{AB}\partial_\sigma\phi'^{*A\mu_3\dots\mu_s}C^\alpha\phi'^B_{\alpha\mu_3\dots\mu_{s-1}}\end{aligned} \tag{5.2}$$

Then, the proof of the inconsistency of this candidate is exactly the same as for spin 4. In fact, for every spin $s \geq 4$, there are only 55 possible terms in the vertex. We have thus managed to adapt the proof to spin s, this deformation is obstructed.

For the second category, the structure has to be $C^* U^{(i)} U^{(j)}$, $i < j$ But C^* carries one index and $U^{(i)}$ carries $i + s - 1$. As no traces can be taken, it is obvious that $i = j - 1$, which leaves us with a family of candidates:

$$a_{2,2j-1} = l_{AB} C^{*\alpha} U^{(j-1)A\mu_1\nu_1|\ldots|\mu_{j-1}\nu_{j-1}|\nu_j\ldots\nu_{s-1}} U^{(j)B}_{\mu_1\nu_1|\ldots|\mu_{j-1}\nu_{j-1}|\alpha\nu_j|\nu_{j+1}\ldots\nu_{s-1}} d^D x$$

Let us now check if these candidates satisfy the equation $\delta a_2 + \gamma a_1 = db_1$ for some a_1. For the second category, we get schematically $\delta a_2 = d(\ldots) + \gamma(\ldots) + l_{AB} h^* U^{(j)A} U^{(j)B} + l_{AB} h^* U^{(j-1)A} U^{(j+1)B}$. The first obstruction can be removed by imposing $l_{AB} = l_{(AB)}$ while the second cannot be removed unless $j = s - 1$. As the tensor $U^{(s)B}$ does not exist, this term is not present at top number of derivatives, the second candidate a_2 that correspond to an a_1 is then:

$$a_{2,2s-3} = l_{AB} C^{*\alpha} U^{(s-2)A\mu_1\nu_1|\ldots|\mu_{s-2}\nu_{s-2}|\nu_{s-1}} U^{(s-1)B}_{\mu_1\nu_1|\ldots|\mu_{s-2}\nu_{s-2}|\alpha\nu_{s-1}} d^D x \,. \tag{5.3}$$

5.2 Exhaustive list of cubic $V^{\Lambda=0}(2, s, s)$ couplings

Using the results of [4], we learn that there exist only *three* cubic couplings of the form $V^{\Lambda=0}(2, s, s)$. They involve a total number of derivatives in the vertex being respectively $2s + 2$, $2s$ and $2s - 2$. Moreover, it is indicated [4] that the $2s$-derivative coupling only exists in dimension $D > 4$. From our results of the last subsection, we conclude that the last coupling is the nonabelian coupling with $2s - 2$ derivatives. The coupling with $2s + 2$ derivatives is simply the strictly-invariant Born-Infeld-like vertex

$$\overset{(3)}{\mathscr{L}}_{BI} = t_{(ab)}\, K^{\alpha\beta|\gamma\delta}\; K^{a\quad\;\mu_1\nu_1|\ldots|\mu_{s-1}\nu_{s-1}}_{\alpha\beta|}\; K^b_{\gamma\delta|\mu_1\nu_1|\ldots|\mu_{s-1}\nu_{s-1}} \,, \tag{5.4}$$

whereas the vertex with $2s$ derivatives is given by

$$\overset{(3)}{\mathscr{L}}_{2s} = u_{(ab)}\, \delta^{[\mu\nu\rho\sigma\lambda]}_{[\alpha\beta\gamma\delta\varepsilon]}\; h^\alpha_{\ \mu}\; K^{a\ \beta\gamma|\quad\ |\mu_1\nu_1|\ldots|\mu_{s-2}\nu_{s-2}}_{\qquad\ \nu\rho}\; K^{b\ \delta\varepsilon|}_{\qquad\sigma\lambda|\mu_1\nu_1|\ldots|\mu_{s-2}\nu_{s-2}} \,. \tag{5.5}$$

It is easy to see that this vertex is not identically zero and is gauge-invariant under the abelian transformations, up to a total derivative.

6. Summary and conclusions

Already for $\Lambda = 0$ the notion of minimal coupling needs to be refined to account for nonabelian vertices with more than two derivatives. Using the antifield formulation [9], in order to prove that the first nonabelian vertex involving a set $\{\phi^i\}$ of fields is cubic, one needs a technical cohomological result concerning the nature of $H_k(\delta|d)$ in the space of invariant polynomials. This technical result has been obtained previously up to $s = 3$ and has been pushed here up to $s = 4$ (cf. appendix B). Supposing that this result holds in the general spin-s case, which is equivalent to supposing that the first nonabelian vertex is cubic, we have shown in section 5 that there exist only two possible nonabelian type $2-s-s$ deformations of the gauge algebra that can be integrated to corresponding gauge transformations. One of these two candidates has $2s-3$ derivatives and must therefore give rise to a vertex with $2s-2$ derivatives to be identified with the flat limit of the corresponding FV $2-s-s$ top vertex [11, 12]. We have shown that the other candidate is obstructed. If liftable to a vertex, it would have given the two-derivatives vertex that corresponds to the minimal Lorentz covariantization. We have thus proved by cohomological methods what has recently been obtained by other methods in [4 − 6].

Then, by combining our cohomological results with those of Metsaev [4], we explicitly built the *exhaustive* list of nontrivial, manifestly covariant vertices $V^{\Lambda=0}(1,s,s)$ and $V^{\Lambda=0}(2,s,s)$, notifying the relevant information concerning the nature of the deformed gauge algebra.

For $\Lambda \neq 0$, the standard notion of Lorentz covariantization does apply although it only provides the bottom vertex of a finite expansion in derivatives covered by inverse powers of Λ, whose top vertices therefore dominate amplitudes (unless extra scales are brought in e.g. by expansions around non-trivial backgrounds). The top-vertices scale with energy non-uniformly for different spins rendering the standard semi-classical approach ill-defined unless some additional feature shows up beyond the cubic level.

Indeed, Vasiliev's fully non-linear higher-spin field equations may provide such a mechanism whereby infinite tails amenable to re-summation are developed. The two parallel perturbative expansions in g and $(\ell\lambda)^{-1}$ resembles those in g_s and $\alpha'\ell^{-2}$ in string theory suggesting that the strong coupling at $(\ell\lambda)^{-1} \gg 1$ corresponds to a tensionless limit of a microscopic string (or membrane).[11] Moreover, the geometric underpinning of Vasiliev's equations is that of flat connections and covariantly constant sections over a base-manifold — the "unfold" — taking their values in a fiber. This suggests that the total system is described by an action integrated over the unfold as well as the phase space of an internal microscopic quantum theory, with kinetic terms given by the sum of the exterior derivative and the internal BRST operator [41]. In this formulation all components of the master fields (including the auxiliary fields) are kept as independent variables, and the problems associated with negative powers of λ are expected to resurface as problematic operator products, although the details of this form of approach remain to be uncovered. A candidate for the microscopic theory is the tensionless string/membrane in AdS whose phase-space action has been argued in [41, 42] to be equivalent to a topological gauged non-compact WZW model with subcritical level [43].

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[11] As is well-known, string theory, whether in flat or anti-de Sitter spacetime (and which one may view as a broken phase of some higher-spin gauge theory), also admits a low-energy limit where the physical energy scale $1/\ell$ becomes much smaller than the mass-scale of the massive fundamental string states, i.e. $\sqrt{\alpha'}/\ell \ll 1$. In this limit the higher-derivative vertices in effective string field theory action become suppressed, leading to an ordinary (non-exotic) higher-derivative field theory.

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Note Added. During the preparation of this manuscript there appeared the work [44] that also addresses the issue of the AdS deformation of the nonabelian 3-3-2 flat-space vertex found in [7].

A. The unique $V^{\Lambda=0}(1,2,2)$ vertex

A.1 The gauge algebra, transformations and vertex: a_2, a_1 and a_0

Thanks to he considerations made above, and as Poincaré invariance is required, the only nontrivial a_2 terms are linear in the undifferentiated *antigh* 2 antifields and quadratic in the non exact ghosts. Family indices can be introduced, which allows to multiply the terms by structure constants. This construction is impossible for 2 spin 1 and 1 spin 2, while there are 3 candidates for 1 spin 1 and 2 spin 2:

- $a_{2,1} = f_{[ab]} C^* C^a_\mu C^{b\mu} d^D x$
- $a_{2,2} = g_{ab} C^{*a}_\mu C C^{b\mu} d^D x$
- $a_{2,3} = l_{[ab]} C^* \partial_{[\mu} C^a_{\nu]} \partial^\mu C^{b\nu} d^D x$

We must now check if the equation $\delta a_2 + \gamma a_1 = d b_1$ admits solutions for the above candidates. Let us note that homogenous solutions for a_1 have to be considered: $\gamma \bar{a}_1 = d\bar{b}_1$. Thanks to Proposition 2, this equation can be redefined as $\gamma \bar{a}_1 = 0$. The non trivial $\bar{a}_1$ are elements of $H(\gamma)$, and, as they are linear in the fields, involve at least one derivative.

$$\begin{aligned} \bullet\ \delta a_{2,1} &= -f_{[ab]} \partial_\rho A^{*\rho} C^a_\mu C^{b\mu} d^D x \\ &= 2 f_{[ab]} A^{*\rho} \partial_\rho C^a_\mu C^{b\mu} d^D x + d(\dots) \\ &= -\gamma (f_{[ab]} A^{*\rho} h^a_{\rho\mu} C^{b\mu} d^D x) + 2 f_{[ab]} A^{*\rho} \partial_{[\rho} C^a_{\mu]} C^{b\mu} d^D x + d(\dots) \,. \end{aligned} \tag{A.1}$$

The second term can not be γ-exact, therefore the first candidate has to be discarded.

$$\begin{aligned} \bullet\ \delta a_{2,2} &= -2 g_{ab} \partial_\nu h^{*a\mu\nu} C C^b_\mu d^D x \\ &= 2 g_{ab} h^{*a\mu\nu} \left[\partial_\nu C C^b_\mu + C \partial_\nu C^b_\mu\right] d^D x + d(\dots) \\ &= -\gamma \left(g_{ab} h^{*a\mu\nu} \left[2 A_\mu C^b_\nu - C h^b_{\mu\nu}\right] d^D x\right) + d(\dots) \,. \end{aligned} \tag{A.2}$$

As there is no homogenous solution with no derivatives, we can conclude that

$$a_{1,2} = g_{ab} h^{*a\mu\nu} \left[2 A_\mu C^b_\nu - C h^b_{\mu\nu}\right] d^D x + \gamma(\dots) \,. \tag{A.3}$$

Finally, applying the Koszul-Tate differential on the $a_{2,3}$ gives

$$\begin{aligned} \bullet\ \delta a_{2,3} &= -l_{[ab]} \partial_\rho A^{*\rho} \partial_{[\mu} C^a_{\nu]} \partial^\mu C^{b\nu} d^D x \\ &= 2 l_{[ab]} A^{*\rho} \partial^2_{\rho[\mu} C^a_{\nu]} \partial^\mu C^{b\nu} d^D x + d(\dots) \\ &= -\gamma \left(2 l_{[ab]} A^{*\rho} \partial_{[\mu} h^a_{\nu]\rho} \partial^\mu C^{b\nu} d^D x \right) + d(\dots) \,. \end{aligned} \tag{A.4}$$

Here we may assume the existence of an homogenous solution:

$$a_{1,3} = \tilde{a}_{1,3} + \bar{a}_{1,3} = 2l_{[ab]} A^{*\rho} \partial_{[\mu} h^a_{\nu]\rho} \partial^\mu C^{b\nu} d^D x + \bar{a}_{1,3} \mid \gamma \bar{a}_{1,3} = 0 \,. \tag{A.5}$$

Finally, we can compute the possible vertices a_0, that have to be a solution of $\delta a_1 + \gamma a_0 = db_0$ where a_1 is one of the above candidates.

For the candidate $a_{1,2}$, we get $\delta a_{1,2} = -2 g_{ab} H^{a\mu\nu} \left[2 A_\mu C^b_\nu - C h^b_{\mu\nu} \right] d^D x$. The second term is γ-exact modulo d if $g_{ab} = g_{[ab]}$ (thanks to the properties of the Einstein tensor), but the first one does not work (terms of the form $h^a_{..} \partial^2_{..} A.C.$ have non vanishing coefficients and are not γ-exact).

Let us now compute the solution $a_{0,3}$, given that $\gamma \partial_{[\mu} h^a_{\nu]\rho} = \partial^2_{\rho[\mu} C^a_{\nu]}$ and $\partial^\rho K^a_{\mu\nu|\rho\sigma} = 2\partial_{[\mu} K^a_{\nu]\sigma}$:

$$\begin{aligned}
\delta \tilde{a}_{1,3} &= 2l_{[ab]} \partial_\sigma F^{\sigma\rho} \partial_{[\mu} h^a_{\nu]\rho} \partial^\mu C^{b\nu} \\
&= 2l_{[ab]} \partial^\rho A^\sigma K^a_{\mu\nu\rho\sigma} \partial^\mu C^{b\nu} + \gamma \left(l_{[ab]} F^{\rho\sigma} \partial_{[\mu} h^a_{\nu]\rho} \partial^\mu h^{b\nu}{}_\sigma \right) + d(\ldots) \\
&= -4 l_{[ab]} A^\rho \partial_{[\mu} K^a_{\nu]\rho} \partial^\mu C^{b\nu} + 4 l_{[ab]} C \partial_{[\mu} K^a_{\nu]\sigma} \partial^\mu h^{b\nu\sigma} \\
&\quad -\gamma \left(2 l_{[ab]} A^\rho K^a_{\mu\nu|\sigma\rho} \partial^\mu h^{b\nu\sigma} - l_{[ab]} F^{\rho\sigma} \partial_{[\mu} h^a_{\nu]\rho} \partial^\mu h^{b\nu}{}_\sigma \right) + d(\ldots) \,.
\end{aligned}$$

The first two terms are δ-exact and correspond to a nontrivial $\bar{a}_{1,3}$. The last two terms are the vertex:

$$\begin{aligned}
a_{0,3} &= l_{[ab]} \left[-F^{\rho\sigma} \partial_{[\mu} h^a_{\nu]\rho} \partial^\mu h^{b\nu}{}_\sigma + 2 A^\sigma K^a_{\mu\nu|\rho\sigma} \partial^\mu h^{b\nu\rho} \right] d^D x \,, \\
\bar{a}_{1,3} &= 2 l_{[ab]} h^{*a\nu\rho} \left[C K^b_{\nu\rho} + F^\mu{}_\rho \partial_{[\mu} C^b_{\nu]} \right] - \frac{1}{D-2} l_{[ab]} h^{*a\prime} \left[C K^b + F^{\mu\nu} \partial_\mu C^b_\nu \right] + \gamma(\ldots)
\end{aligned} \tag{A.6}$$

A.2 Inconsistency with Einstein-Hilbert theory

Here we show that, as expected, the spin-2 massless fields considered in the previous section cannot be considered as the linearized Einstein-Hilbert graviton. Let us consider the second order in g in the master equation: it can be written $(W_1, W_1) = -2sW_2$. Let us decompose W_2 according to the antighost number: $W_2 = \int (c_0 + c_1 + c_2 + \ldots)$. We will just check here the highest antighost part: $(a_2, a_2) = -2\gamma c_2 - 2\delta c_3 + d(\ldots)$. But indeed, in any theory in which a_2 is linear in the antighost 2 antifields and quadratic in the ghosts, (a_2, a_2) cannot depend on the antighost 1 antifields or on the fields, so that no δc_3 can appear. This indicates that the expansion of W_2 stops at antighost 2 for those theories. But in fact, we get here:

$$(a_{2,3}, a_{2,3}) = 2 \frac{\delta a_{2,3}}{\delta C^{*\mu}_a} \frac{\delta a_{2,3}}{\delta C^a_\mu} + 2 \frac{\delta a_{2,3}}{\delta C^*} \frac{\delta a_{2,3}}{\delta C} = 0 \quad .$$

This just means that the solution that we found is self-consistent at that order. But we also have to check the compatibility with self-interacting spin 2 fields. Let us consider the a_2 for a collection of Einstein-Hilbert theories (this can be found in [34]): $a_{2,EH} = f_{(abc)} C^{*a\mu} C^{b\nu} \partial_{[\mu} C^c_{\nu]}$, in which the coefficients f_{abc} can be chosen diagonal. Let us now compute

$$\begin{aligned}
(a_{2,EH}, a_{2,3}) &= \frac{\delta a_{2,EH}}{\delta C^{*\rho}_e} \frac{\delta a_{2,3}}{\delta C^e_\rho} = -2 f^e_{bc} l_{ea} C^{b\nu} \partial_{[\rho} C^c_{\nu]} \partial_\tau \left[C^* \partial^{[\tau} C^{|a|\rho]} \right] \\
&= \gamma(\ldots) + d(\ldots) - 2 f^e_{bc} l_{ae} C^* \eta^{\sigma\nu} \partial_{[\tau} C^b_{\sigma]} \partial_{[\rho} C^c_{\nu]} \partial^{[\tau} C^{|a|\rho]} \,.
\end{aligned} \tag{A.7}$$

This can be consistent only if $f^e_{(bc}l_{a)e} = 0$. But if we choose f_{abc} diagonal, we obtain $f^a_{aa}l_{ba} = -2f^a_{ab}l_{aa} = 0$, which means that the f's or the l's have to vanish. In other words, the spin 2 particles interacting with the spin 1 in our vertex cannot be Einstein-Hilbert gravitons.

B. Invariant cohomology of δ modulo d for spin 4

The following theorem is crucial, in the sense that it enables one to prove the uniqueness of the deformations, within the cohomological approach of [9]:

Theorem 2. *Assume that the invariant polynomial a^p_k (p = form-degree, k = antifield number) is δ-trivial modulo d,*

$$a^p_k = \delta\mu^p_{k+1} + d\mu^{p-1}_k \quad (k \geqslant 2). \tag{B.1}$$

Then, one can always choose μ^p_{k+1} and μ^{p-1}_k to be invariant.

To prove the theorem, we need the following lemma, proved in [34].

Lemma 1. *If a is an invariant polynomial that is δ-exact, $a = \delta b$, then, a is δ-exact in the space of invariant polynomials. That is, one can take b to be also invariant.*

The proof of theorem 2 for spin-4 gauge field proceeds in essentially the same way as for the spin-3 case presented in detail in [35], to which we refer for the general lines of reasoning. We only give here the piece of proof where things differ significantly from the spin-3 case.

Different situations are considered, depending on the values of p and k. In form degree $p < D$, the proof goes as in [35]. In form degree $p = D$, two cases must be considered: $k > D$ and $k \leqslant D$. In the first case, the proof goes as in [35], the new features appearing when $p = D$ and $k \leqslant D$. Rewriting the top equation (i.e. (B.1) with $p = D$) in dual notation, we have

$$a_k = \delta b_{k+1} + \partial_\rho j^\rho_k, \;\; (k \geqslant 2). \tag{B.2}$$

We will work by induction on the antifield number, showing that if the property expressed in theorem 2 is true for $k+1$ (with $k > 1$), then it is true for k. As we already know that it is true in the case $k > D$, the theorem will be proved.

Inductive proof for $k \leqslant D$. The proof follows the lines of ref. [39] and decomposes in two parts. First, all Euler-Lagrange derivatives of (B.2) are computed. Second, the Euler-Lagrange (E.L.) derivative of an invariant quantity is also invariant. This property is used to express the E.L. derivatives of a_k in terms of invariants only. Third, the homotopy formula is used to reconstruct a_k from his E.L. derivatives. This almost ends the proof.

(i) Let us take the E.L. derivatives of (B.2). Since the E.L. derivatives with respect to $C^*_{\alpha\beta\gamma}$, the antifield associated with the ghost $C^{\alpha\beta\gamma}$, commute with δ, we get first:

$$\frac{\delta^L a_k}{\delta C^*_{\alpha\beta\gamma}} = \delta Z^{\alpha\beta\gamma}_{k-1} \tag{B.3}$$

with $Z^{\alpha\beta\gamma}_{k-1} = \frac{\delta^L b_{k+1}}{\delta C^*_{\alpha\beta\gamma}}$. For the E.L. derivatives of b_{k+1} with respect to $h^*_{\mu\nu\rho\sigma}$ we obtain, after a direct computation,

$$\frac{\delta^L a_k}{\delta h^*_{\mu\nu\rho\sigma}} = -\delta X^{\mu\nu\rho\sigma}_k + 4\partial^{(\mu} Z^{\nu\rho\sigma)}_{k-1}. \tag{B.4}$$

where $X^{\mu\nu\rho\sigma}_k = \frac{\delta^L b_{k+1}}{\delta h^*_{\mu\nu\rho\sigma}}$. Finally, let us compute the E.L. derivatives of a_k with respect to the fields. We get:

$$\frac{\delta^L a_k}{\delta h_{\mu\nu\rho\sigma}} = \delta Y^{\mu\nu\rho\sigma}_{k+1} + \mathscr{G}^{\mu\nu\rho\sigma|\alpha\beta\gamma\delta} X_{\alpha\beta\gamma\delta|k} \tag{B.5}$$

where $Y^{\mu\nu\rho\sigma}_{k+1} = \frac{\delta^L b_{k+1}}{\delta h_{\mu\nu\rho\sigma}}$ and $\mathscr{G}^{\mu\nu\rho\sigma|\alpha\beta\gamma\delta}(\partial)$ is the second-order self-adjoint differential operator appearing in Fronsdal's equations of motion $0 = \frac{\delta S^F[h]}{\delta h_{\mu\nu\rho\sigma}} \equiv G^{\mu\nu\rho\sigma} = \mathscr{G}^{\mu\nu\rho\sigma|\alpha\beta\gamma\delta}\, h_{\alpha\beta\gamma\delta}$. The hermiticity of $\mathscr{G}$ implies $\mathscr{G}^{\mu\nu\rho\sigma|\alpha\beta\gamma\delta} = \mathscr{G}^{\alpha\beta\gamma\delta|\mu\nu\rho\sigma}$.

(ii) The E.L. derivatives of an invariant object are invariant. Thus, $\frac{\delta^L a_k}{\delta C^*_{\alpha\beta\gamma}}$ is invariant. Therefore, by Lemma 1 and eq. (B.3), we have also

$$\frac{\delta^L a_k}{\delta C^*_{\alpha\beta\gamma}} = \delta Z'^{\alpha\beta\gamma}_{k-1} \tag{B.6}$$

for some invariant $Z'^{\alpha\beta\gamma}_{k-1}$. Indeed, let us write the decomposition $Z^{\alpha\beta\gamma}_{k-1} = Z'^{\alpha\beta\gamma}_{k-1} + \tilde{Z}^{\alpha\beta\gamma}_{k-1}$, where $\tilde{Z}^{\alpha\beta\gamma}_{k-1}$ is obtained from $Z^{\alpha\beta\gamma}_{k-1}$ by setting to zero all the terms that belong only to $H(\gamma)$. The latter operation clearly commutes with taking the δ of something, so that eq. (B.3) gives $0 = \delta\tilde{Z}^{\alpha\beta\gamma}_{k-1}$ which, by the acyclicity of δ, yields $\tilde{Z}^{\alpha\beta\gamma}_{k-1} = \delta\sigma^{\alpha\beta\gamma}_k$ where $\sigma^{\alpha\beta\gamma}_k$ can be chosen to be traceless. Substituting $\delta\sigma^{\alpha\beta\gamma}_k + Z'^{\alpha\beta\gamma}_{k-1}$ for $Z^{\alpha\beta\gamma}_{k-1}$ in eq. (B.3) gives eq. (B.6).

Similarly, one easily verifies that

$$\frac{\delta^L a_k}{\delta h^*_{\mu\nu\rho\sigma}} = -\delta X'^{\mu\nu\rho\sigma}_k + 4\partial^{(\mu} Z'^{\nu\rho\sigma)}_{k-1}\,, \tag{B.7}$$

where $X^{\mu\nu\rho\sigma}_k = X'^{\mu\nu\rho\sigma}_k + 4\partial^{(\mu}\sigma^{\nu\rho\sigma)}_k + \delta\rho^{\mu\nu\rho\sigma}_{k+1}$. Finally, using $\mathscr{G}^{\mu\nu\rho\sigma}{}_{\alpha\beta\gamma\delta}\,\partial^{(\alpha}\sigma^{\beta\gamma\delta)}{}_k = 0$ due to the gauge invariance of the equations of motion ($\sigma_{\alpha\beta\delta}$ has been taken traceless), we find

$$\frac{\delta^L a_k}{\delta h_{\mu\nu\rho\sigma}} = \delta Y'^{\mu\nu\rho\sigma}_{k+1} + \mathscr{G}^{\mu\nu\rho\sigma}{}_{\alpha\beta\gamma\delta} X'^{\alpha\beta\gamma\delta}_k \tag{B.8}$$

for the invariants $X'^{\mu\nu\rho\sigma}_k$ and $Y'^{\mu\nu\rho\sigma}_{k+1}$. Before ending the argument by making use of the homotopy formula, it is necessary to know more about the invariant $Y'^{\mu\nu\rho\sigma}_{k+1}$.

Since a_k is invariant, it depends on the fields only through the curvature K, the Fronsdal tensor and their derivatives. (We substitute $4\,\partial^{[\delta}\partial_{[\gamma}F_{\rho]}{}^{\sigma]}{}_{\mu\nu}$ for $\eta^{\alpha\beta}K^{\delta\sigma}{}_{|\alpha\mu|\beta\nu|\gamma\rho}$ everywhere.) We then express the Fronsdal tensor in terms of the Einstein tensor:

$F_{\mu\nu\rho\sigma} = G_{\mu\nu\rho\sigma} - \frac{6}{n+2}\,\eta_{(\mu\nu}G_{\rho\sigma)}$, so that we can write $a_k = a_k([\Phi^{*i}],[K],[G])$, where $[G]$ denotes the Einstein tensor and its derivatives. We can thus write

$$\frac{\delta^L a_k}{\delta h_{\mu\nu\rho\sigma}} = \mathscr{G}^{\mu\nu\rho\sigma}{}_{\alpha\beta\gamma\delta}A'^{\alpha\beta\gamma\delta}_k + \partial_\alpha\partial_\beta\partial_\gamma\partial_\delta M'^{\alpha\mu|\beta\nu|\gamma\rho|\delta\sigma}_k \tag{B.9}$$

where

$$A'^{\alpha\beta\gamma\delta}_k \propto \frac{\delta a_k}{\delta G_{\alpha\beta\gamma\delta}}$$

and

$$M'^{\alpha\mu|\beta\nu|\gamma\rho|\delta\sigma}_k \propto \frac{\delta a_k}{\delta K_{\alpha\mu|\beta\nu|\gamma\rho|\delta\sigma}}$$

are both invariant and respectively have the same symmetry properties as the "Einstein" and "Riemann" tensors.

Combining eq. (B.8) with eq. (B.9) gives

$$\delta Y'^{\mu\nu\rho\sigma}_{k+1} = \partial_\alpha\partial_\beta\partial_\gamma\partial_\delta M'^{\alpha\mu|\beta\nu|\gamma\rho|\delta\sigma}_k + \mathscr{G}^{\mu\nu\rho\sigma}{}_{\alpha\beta\gamma\delta}B'^{\alpha\beta\gamma\delta}_k \tag{B.10}$$

with $B'^{\alpha\beta\gamma\delta}_k := A'^{\alpha\beta\gamma\delta}_k - X'^{\alpha\beta\gamma\delta}_k$. Now, only the first term on the right-hand-side of eq. (B.10) is divergence-free, $\partial_\mu(\partial_{\alpha\beta\gamma}M'^{\alpha\mu|\beta\nu|\gamma\rho}_k) \equiv 0$, not the second one which instead obeys a relation analogous to the Noether identities

$$\partial^\tau G_{\mu\nu\rho\tau} - \frac{3}{(n+2)}\,\eta_{(\mu\nu}\partial^\tau G'_{\rho)\tau} = 0\,.$$

As a result, we have $\delta\Big[\partial_\mu(Y'^{\mu\nu\rho\sigma}_{k+1} - \frac{3}{D+2}\,\eta^{(\nu\rho}Y'^{\sigma)\mu}_{k+1})\Big] = 0$, where $Y'^{\mu\sigma}_{k+1} \equiv \eta_{\nu\rho}Y'^{\mu\nu\rho\sigma}_{k+1}$. By Lemma 1, we deduce

$$\partial_\mu\Big(Y'^{\mu\nu\rho\sigma}_{k+1} - \frac{3}{D+2}\,\eta^{(\nu\rho}Y'^{\sigma)\mu}_{k+1}\Big) + \delta F'^{\nu\rho\sigma}_{k+2} = 0\,, \tag{B.11}$$

where $F'^{\nu\rho\sigma}_{k+2}$ is invariant and can be chosen symmetric and traceless. Eq. (B.11) determines a cocycle of $H^{D-1}_{k+1}(d|\delta)$, for given ν, ρ and σ . Using the general isomorphisms $H^{D-1}_{k+1}(d|\delta) \cong H^D_{k+2}(\delta|d) \cong 0$ $(k \geqslant 1)$ [37] we deduce

$$Y'^{\mu\nu\rho\sigma}_{k+1} - \frac{3}{D+2}\,\eta^{(\nu\rho}Y'^{\sigma)\mu}_{k+1} = \partial_\alpha T^{\alpha\mu|\nu\rho\sigma}_{k+1} + \delta P^{\mu\nu\rho\sigma}_{k+2}\,, \tag{B.12}$$

where both $T^{\alpha\mu|\nu\rho\sigma}_{k+1}$ and $P^{\mu\nu\rho\sigma}_{k+2}$ are invariant by the induction hypothesis. Moreover, $T^{\alpha\mu|\nu\rho\sigma}_{k+1}$ is antisymmetric in its first two indices. The tensors $T^{\alpha\mu|\nu\rho\sigma}_{k+1}$ and $P^{\mu\nu\rho\sigma}_{k+2}$ are both symmetric and traceless in (ν,ρ,σ). This results easily from taking the trace of eq. (B.12) with $\eta_{\nu\rho}$ and using the general isomorphisms $H^{D-2}_{k+1}(d|\delta) \cong H^{D-1}_{k+2}(\delta|d) \cong H^D_{k+3}(\delta|d) \cong 0$ [37] which hold since k is positive. From eq. (B.12) we obtain

$$Y'^{\mu\nu\rho\sigma}_{k+1} = \partial_\alpha[T^{\alpha\mu|\nu\rho\sigma}_{k+1} + \frac{3}{D}\,T^{\alpha|\mu(\nu}_{k+1}\eta^{\rho\sigma)}] + \delta(\ldots) \tag{B.13}$$

where $T^{\alpha|\mu\nu}_{k+1} \equiv \eta_{\tau\rho} T^{\alpha\tau|\rho\mu\nu}_{k+1}$. We do not explicit the δ-exact term since it plays no role in the following. Since $Y'^{\mu\nu\rho\sigma}_{k+1}$ is symmetric in μ and ν, we have also

$$\partial_\alpha \left(T^{\alpha[\mu|\nu]}_{k+1}{}_{\rho\sigma} + \frac{2}{D}\, T^{\alpha|[\mu}_{k+1}{}_{(\sigma}\delta^{\nu]}_{\rho)} \right) + \ \delta(\dots) = 0\,.$$

The triviality of $H^D_{k+2}(d|\delta)$ $(k > 0)$ implies again that $T^{\alpha[\mu|\nu]}_{k+1}{}_{\rho\sigma} + \frac{2}{D}\, T^{\alpha|[\mu}_{k+1}{}_{(\sigma}\delta^{\nu]}_{\rho)}$ is trivial, in particular,

$$\partial_\beta S'^{\beta\alpha|\mu\nu|}{}_{\rho\sigma} + \delta(\dots) = T^{\alpha[\mu|\nu]}_{k+1}{}_{\rho\sigma} + \frac{2}{D}\, T^{\alpha|[\mu}_{k+1}{}_{(\sigma}\delta^{\nu]}_{\rho)} \tag{B.14}$$

where $S'^{\beta\alpha|\mu\nu|}{}_{\rho\sigma}$ is antisymmetric in the pairs of indices (β, α) and (μ, ν), while it is symmetric and traceless in (ρ, σ). Actually, it is traceless in $\mu, \nu, \rho\, \sigma$ as the right-hand side of the above equation shows. The induction assumption allows us to choose $S'^{\beta\alpha|\mu\nu|}{}_{\rho\sigma}$, as well as the quantity under the Koszul-Tate differential δ . We now project both sides of eq. (B.14) on the following irreducible representation of the orthogonal group $\begin{array}{|c|c|c|}\hline \alpha&\mu&\sigma\\ \hline \rho&\nu \\ \cline{1-2}\end{array}$ and obtain

$$\partial_\beta W'^{\beta|\alpha\rho|\mu\nu|\sigma}_{k+1} + \delta(\dots) = 0 \tag{B.15}$$

where $W'^{\beta|\alpha\rho|\mu\nu|\sigma}_{k+1}$ denotes the corresponding projection of $S'^{\beta\alpha|\mu\nu|\rho\sigma}$. Eq. (B.15) determines, for given $(\mu, \nu, \alpha, \rho, \sigma)$, a cocycle of $H^{D-1}_{k+1}(d|\delta, H(\gamma))$. Using again the isomorphisms [37] $H^{D-1}_{k+1}(d|\delta) \cong H^D_{k+2}(\delta|d) \cong 0$ $(k \geqslant 1)$ and the induction hypothesis, we find

$$W'^{\beta|\alpha\rho|\mu\nu|\sigma}_{k+1} = \partial_\lambda \phi^{\lambda\beta|\alpha\rho|\mu\nu|\sigma}_{k+1} + \delta(\dots) \tag{B.16}$$

where $\phi^{\lambda\beta|\alpha\rho|\mu\nu|\sigma}_{k+1}$ is invariant, antisymmetric in (λ, β) and possesses the irreducible, totally traceless symmetry $\begin{array}{|c|c|c|}\hline \alpha&\mu&\sigma\\ \hline \rho&\nu \\ \cline{1-2}\end{array}$ in its last five indices. The δ-exact term is invariant as well. Then, projecting the equation (B.16) on the totally traceless irreducible representation $\begin{array}{|c|c|c|}\hline \alpha&\mu&\sigma\\ \hline \rho&\nu&\beta \\ \hline\end{array}$ and taking into account that $W'^{\beta|\alpha\rho|\mu\nu|\sigma}{}_{k+1}$ is built out from $S'^{\beta\alpha|\mu\nu|\rho\sigma}$, we find

$$\partial_\lambda \Psi'^{\lambda|\alpha\rho|\mu\nu|\sigma\beta}_{k+1} + \delta(\dots) = 0 \tag{B.17}$$

where $\Psi'^{\lambda|\alpha\rho|\mu\nu|\sigma\beta}_{k+1}$ denotes the corresponding projection of $\phi^{\lambda\beta|\alpha\rho|\mu\nu|\sigma}_{k+1}$. The same arguments used before imply

$$\Psi'^{\lambda|\alpha\rho|\mu\nu|\sigma\beta}_{k+1} = \partial_\tau \Xi'^{\tau\lambda|\alpha\rho|\mu\nu|\sigma\beta} + \delta(\dots) \tag{B.18}$$

where the symmetries of $\Xi'^{\tau\lambda|\alpha\rho|\mu\nu|\sigma\beta}$ on its last 6 indices can be read off from the left-hand side and where the first pair of indices is antisymmetric. Again, $\Xi'^{\tau\lambda|\alpha\rho|\mu\nu|\sigma\beta}$ can be taken to be invariant.

Then, we take the projection of $\Xi'^{\tau\lambda|\alpha\rho|\mu\nu|\sigma\beta}$ on the irreducible representation $\begin{array}{|c|c|c|c|}\hline \tau&\alpha&\mu&\sigma\\ \hline \lambda&\rho&\nu&\beta \\ \hline\end{array}$ of $GL(D)$ (here we do not impose tracelessness) and denote the result by $\Theta'^{\tau\lambda|\alpha\rho|\mu\nu|\sigma\beta}$.

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This invariant tensor possesses the algebraic symmetries of the invariant spin-4 curvature tensor. Finally, putting all the previous results together, we obtain the following relation, using the symbolic manipulation program *Ricci* [18]:

$$6\,Y'^{\mu\nu\rho\sigma}_{k+1} = \partial_\alpha\partial_\beta\partial_\gamma\partial_\delta\Theta'^{\alpha\mu|\beta\nu|\gamma\rho|\delta\sigma}_{k+1} + \mathscr{G}^{\mu\nu\rho\sigma}{}_{\alpha\beta\gamma\delta}\widehat{X}'^{\alpha\beta\gamma\delta}_{k+1} + \delta(\dots)\,, \tag{B.19}$$

with

$$\begin{aligned}\widehat{X}'_{\alpha\beta\gamma\delta|k+1} := \frac{\mathscr{Y}^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}}{D-2}\Big[& -\frac{1}{3}\,\eta^{\tau\lambda}S_{\tau\mu|\lambda\nu|\rho\sigma|k+1} \\ & +\frac{1}{3(D+1)}\,\eta_{\mu\nu}\eta^{\tau\lambda}\eta^{\kappa\zeta}(S_{\tau\kappa|\lambda\zeta|\rho\sigma|k+1} + 2\,S_{\tau\kappa|\lambda\rho|\zeta\sigma|k+1}) \\ & +\frac{2(D-2)}{D}\,\eta^{\kappa\tau}\partial^\lambda\phi_{\kappa\mu|\lambda\nu|\tau\rho|\sigma} - \frac{4(D-2)}{D(D+2)}\,\eta_{\mu\nu}\eta^{\kappa\tau}\eta^{\xi\zeta}\partial^\lambda\phi_{\kappa\xi|\tau\zeta|\lambda\mu|\nu}\Big]\end{aligned} \tag{B.20}$$

being double-traceless and where $\mathscr{Y}^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}$ projects on completely symmetric rank-4 tensors.

(iii) We can now complete the argument. The homotopy formula

$$a_k = \int_0^1 dt\,\left[C^*_{\alpha\beta\gamma}\frac{\delta^L a_k}{\delta C^*_{\alpha\beta\gamma}} + h^*_{\mu\nu\rho\sigma}\frac{\delta^L a_k}{\delta h^*_{\mu\nu\rho\sigma}} + h^{\mu\nu\rho\sigma}\frac{\delta^L a_k}{\delta h^{\mu\nu\rho\sigma}}\right](th\,,\,th^*\,,\,tC^*) \tag{B.21}$$

enables one to reconstruct a_k from its Euler-Lagrange derivatives. Inserting the expressions (B.6)–(B.8) for these E.L. derivatives, we get

$$a_k = \delta\Big(\int_0^1 dt\,[C^*_{\alpha\beta\gamma}Z'^{\alpha\beta\gamma}_{k-1} + h^*_{\mu\nu\rho\sigma}X'^{\mu\nu\rho\sigma}_k + h_{\mu\nu\rho\sigma}Y'^{\mu\nu\rho\sigma}_{k+1}](t)\Big) + \partial_\rho k^\rho. \tag{B.22}$$

The first two terms in the argument of δ are manifestly invariant. In order to prove that the third term can be assumed to be invariant in eq. (B.22), we use eq. (B.19) to find that (absorbing the irrelevant factor 6 in a redefinition of $Y'^{\mu\nu\rho\sigma}$)

$$h_{\mu\nu\rho\sigma}\,Y'^{\mu\nu\rho\sigma}_{k+1} = \frac{1}{16}\,\Theta'^{\alpha\mu|\beta\nu|\gamma\rho|\delta\sigma}_{k+1}K_{\alpha\mu|\beta\nu|\gamma\rho|\delta\sigma} + G_{\alpha\beta\gamma\delta}\widehat{X}'^{\alpha\beta\gamma\delta}{}_{k+1} + \partial_\rho\ell^\rho + \delta(\dots)\,,$$

where we integrated by part four times in order to get the first term of the r.h.s. while the hermiticity of $\mathscr{G}^{\mu\nu\rho\sigma|\alpha\beta\gamma\delta}$ was used to obtain the second term.

We are left with $a_k = \delta\mu_{k+1} + \partial_\rho\nu^\rho_k$, where μ_{k+1} is invariant. That ν^ρ_k can now be chosen invariant is straightforward. Acting with γ on the last equation yields $\partial_\rho(\gamma\nu^\rho_k) = 0$. By the Poincaré lemma, $\gamma\nu^\rho_k = \partial_\sigma(\tau^{[\rho\sigma]}_k)$. Furthermore, Proposition 2 concerning $H(\gamma|\,d)$ at positive antighost number k implies that one can redefine ν^ρ_k by the addition of trivial d-exact terms such that one can assume $\gamma\nu^\rho_k = 0$. As the pureghost number of ν^ρ_k vanishes, the last equation implies that ν^ρ_k is an invariant polynomial.

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